

Quantum mass and central charge of supersymmetric monopoles: anomalies, current renormalization, and surface terms

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ABSTRACT: We calculate the one-loop quantum corrections to the mass and central charge of $N = 2$ and $N = 4$ supersymmetric monopoles in 3+1 dimensions. The corrections to the $N = 2$ central charge are finite and due to an anomaly in the conformal central charge current, but they cancel for the $N = 4$ monopole. For the quantum corrections to the mass we start with the integral over the expectation value of the Hamiltonian density, which we show to consist of a bulk contribution which is given by the familiar sum over zero-point energies, as well as surface terms which contribute nontrivially in the monopole sector. The bulk contribution is evaluated through index theorems and found to be nonvanishing only in the $N = 2$ case. The contributions from the surface terms in the Hamiltonian are cancelled by infinite composite operator counterterms in the $N = 4$ case, forming a multiplet of improvement terms. These counterterms are also needed for the renormalization of the central charge. However, in the $N = 2$ case they cancel, and both the improved and the unimproved current multiplet are finite.

KEYWORDS: Supersymmetric gauge theory, Anomalies in Field and String Theories, Solitons Monopoles and Instantons, Renormalization Regularization and Renormalons.

Contents

1. Introduction	1
2. The $N = 2$ and $N = 4$ susy monopole models and their supercurrents	4
2.1 $N = 2$	4
2.2 $N = 4$	8
3. Fluctuations about the monopole background and renormalization	10
3.1 Fluctuation equations and propagators	11
3.2 One-loop counterterms	13
4. Bulk and surface contributions to the quantum mass of susy monopoles	15
5. Index theorems and susy spectral densities	19
5.1 The susy kink	25
5.2 The $N = 2$ susy vortex	25
5.3 The $N = 2$ susy monopole	27
6. Results for the quantum monopole mass	28
6.1 Bulk contributions	28
6.2 Surface contributions	29
7. Central charge	32
8. Conclusions	35

1. Introduction

The existence of supersymmetric (susy) monopoles [1, 2] which saturate the Bogomolnyi bound [3] also at the quantum level [4] plays an important role in the successes of nonperturbative studies of super Yang-Mills theories through dualities [5–8].

On the other hand, a direct calculation of quantum corrections to the mass and central charge of susy solitons has proved to be fraught with difficulties and surprises. While in the earliest literature it was assumed that supersymmetry would lead to a complete cancellation of quantum corrections to both [1, 2, 9], it was quickly realized that the bosonic and fermionic quantum fluctuations do not only not cancel, but have to match the infinities in standard coupling and field renormalizations [10–15]. However, even in the simplest case of the 1+1 dimensional minimally supersymmetric kink, there was until the end of the 1990’s an unresolved discrepancy in the literature as to the precise value of one-loop contributions once the renormalization scheme has duly been fixed. As pointed out in [16], most workers

had used regularization methods which when used naively give inconsistent results already for the exactly solvable sine-Gordon kink. In the susy case, there is moreover an extra complication in that the traditionally employed periodic boundary conditions lead to a contamination of the results by energy located at the boundary of the quantization volume, and the issue of the correct quantum mass of the susy kink was finally settled in ref. [17] by the use of topological boundary conditions, which avoid this contamination.¹ This singled out as correct the earlier result of ref. [10, 20] and refuted the null results of refs. [11–14]. However it led to a new problem because it seemed that the central charge did not appear to receive corresponding quantum corrections [15], which would imply a violation of the Bogomolnyi bound. In ref. [17] it was conjectured that a new kind of anomaly was responsible, and in ref. [21] Shifman, Vainstein, and Voloshin subsequently demonstrated that supersymmetry requires an anomalous contribution to the central charge current.² The latter appears in the same multiplet as the trace and conformal-susy anomalies, and ensures BPS saturation even in the $N = 1$ susy kink, where initially standard multiplet shortening arguments seemed not to be applicable.³

In ref. [26] we have developed a version of dimensional regularization which can be used for solitons (and instantons). The soliton is embedded in a higher-dimensional space by adding extra trivial dimensions [27, 28] and choosing a model which is supersymmetric in the bigger space and which reproduces the original model by dimensional reduction. This is thus a combination of standard 't Hooft-Veltman dimensional regularization [29] which goes up in dimensions, and susy-preserving dimensional reduction [30, 31], which goes down. In ref. [24] we demonstrated how the anomalous contribution to the central charge of the susy kink can be obtained as a remnant of parity violation in the odd-dimensional model used for embedding the susy kink, and recently we showed that the same kind of anomalous contribution arises in the more prominent case of the 3+1-dimensional monopole of $N = 2$ super-Yang-Mills theory in the Higgs⁴ phase [35]. This previously overlooked [36, 37] finite contribution turns out to be in fact essential for consistency of these direct calculations with the $N = 2$ low-energy effective action of Seiberg and Witten [5–7]. (We have also found previously overlooked finite contributions to both mass and central charge of the $N = 2$ vortex in 2+1 dimensions [38, 39], which are however not associated with conformal anomalies but are rather standard renormalization effects.)

In the case of the $N = 4$ monopole, it turns out that the quantum corrections to the mass and central charge are anomaly-free in accordance with the finiteness of the model

¹Ref. [17] used “derivative regularization” to make this work. In mode regularization it turns out that one has to average over sets of boundary conditions to cancel both localized boundary energy and delocalized momentum [18, 19].

²Refs. [22, 23], who had obtained the correct value for the quantum mass also claimed a nontrivial quantum correction to the central charge apparently without the need of the anomalous term proposed in ref. [21]. However, as shown in ref. [24], this was achieved by formal arguments handling ill-defined since unregularized quantities.

³That multiplet shortening also occurs in the $N = 1$ susy kink was eventually clarified in ref. [25].

⁴Anomalous contributions to the central charge appear also in the newly discovered “confined monopoles” pertaining to the Coulomb phase [32, 33], which turn out to be related to central charge anomalies of 1+1-dimensional $N = 2$ sigma models with twisted mass [34].

and the vanishing of the trace and conformal-susy anomalies. However, a direct calculation of these corrections leads to the surprising appearance of composite-operator counterterms that were absent in the $N = 2$ case. This issue was recently clarified in ref. [40], where it was found that in the 3+1-dimensional cases there is generally a need for composite-operator counterterms forming a multiplet of improvement terms, except when $N = 2$.

In this paper we give a unified treatment of the $N = 2$ and $N = 4$ monopoles and present direct and complete one-loop calculations of anomalous and nonanomalous contributions to their quantum mass and central charge. This requires a careful calculation of both bulk contributions and surface terms. The anomalous contributions arise from the bulk, which in the case of the mass are given by sums over zero-point energies. In the central charge, on the other hand, such bulk contributions appear through momentum operators in the extra dimensions introduced by our method of dimensional regularization. Both types of bulk contributions can be evaluated through the use of index theorems, which express nontrivial differences for the spectral densities of bosonic and fermionic contributions in terms of the axial anomaly [41, 42, 37]. Surface terms, which at the classical level yield both the central charge and the mass, can also produce quantum corrections to the mass and the central charge in the monopole sector, in stark contrast to the situation in lower-dimensional models (which do not involve massless fields). The traditional classical value of the monopole mass and central charge is obtained when (and only when) unimproved currents are used, which are singled out when the $N = 2$ and $N = 4$ super-Yang-Mills theories are derived from dimensional reduction of the $N = 1$ super-Yang-Mills theory in 5+1 and 9+1 dimensions, respectively. Concerning quantum corrections due to the surface terms, the $N = 2$ monopole is special in that both unimproved and improved currents are finite, but in the $N = 4$ monopole the unimproved current multiplet requires additive infinite composite-operator renormalization through a multiplet of improvement terms.

After setting up the models in question in section 2 and presenting their solitonic solutions, the susy algebra and the associated currents, we discuss quantization, regularization and renormalization of these models in section 3. In section 4 we start our direct calculation of the monopole mass by taking the integral over the expectation value of the energy density, and separating bulk and surface contributions so that the former correspond to the familiar sums over zero-point energies. In section 5 we derive and discuss the index theorems for evaluating the former [42, 43], and apply them to all lower-dimensional models as well as the monopole and instantons. In section 6 we evaluate the quantum corrections to the mass arising from the surface contributions. These turn out to cancel in the $N = 2$ case, but require infinite renormalization of the currents in the otherwise finite $N = 4$ theory. By combining the anomalous and nonanomalous contributions we obtain the final result for the $N = 2$ and $N = 4$ quantum mass. In section 7, we perform the analogous direct evaluation of the expectation value of the central charges. Here the anomalous contribution of the $N = 2$ model is identified as the remnant of parity violation in the 4+1-dimensional theory that one can use for trivial embedding of a monopole as a string-like object without violating supersymmetry. Section 8 contains our conclusions and some assorted comments on technical issues encountered in calculations of the quantum corrections to mass and central charge of susy solitons.

2. The $N = 2$ and $N = 4$ susy monopole models and their supercurrents

In this section we discuss the $N = 2$ and $N = 4$ super Yang-Mills models with monopoles, their BPS equations, susy algebra, improvement terms and the susy variation of the latter. We begin with the $N = 2$ case, and then present the $N = 4$ case.

2.1 $N = 2$

The $N = 2$ super Yang-Mills theory in 3+1 dimensions can be obtained by dimensional reduction from the simplest ($N = 1$) super Yang-Mills theory in 5+1 dimensions [44]. Instead of using two complex chiral spinors with a symplectic Majorana condition in order to exhibit the R symmetry group $U(2)$ of the $N = 2$ susy algebra in 3+1 dimensions [45], we use the simpler formulation with one complex chiral Dirac field λ . The Lagrangian in 5+1 dimensions reads

$$\mathcal{L} = -\frac{1}{4}F_{MN}^a F^{aMN} - \bar{\lambda}^a \Gamma^M (D_M \lambda)^a, \quad (2.1)$$

where the indices M, N take the values $0, 1, 2, 3, 5, 6$, and the metric has signature $(-, +, \dots, +)$. The Dirac matrices are 8×8 matrices, and only Γ^0 is anti-hermitian. Defining $\bar{\lambda}^a = (\lambda^a)^\dagger i \Gamma^0$, the action is hermitian without an extra factor of i in front of the fermionic part (up to fermionic surface terms which do not contribute, not even in the solitonic sector). The covariant derivative $(D_M \lambda)^a$ is defined by $\partial_M \lambda^a + g f^{abc} A_M^b \lambda^c$. We use $SU(2)$ as gauge group with $f^{abc} = \epsilon^{abc}$. Since λ^a is in the adjoint representation, there is no difference between $\bar{\lambda}^a$ and $\bar{\lambda}_a$.

The chirality condition can be written as

$$(1 - \Gamma_7) \lambda = 0 \quad \text{with} \quad \Gamma_7 = \Gamma_0 \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_5 \Gamma_6 \quad (2.2)$$

and $\Gamma_7^2 = 1$. To carry out the dimensional reduction we write $A_M = (A_\mu, P, S)$ and choose the following representation of gamma matrices

$$\begin{aligned} \Gamma_\mu &= \gamma_\mu \otimes \sigma_1, & \mu &= 0, 1, 2, 3, \\ \Gamma_5 &= \gamma_5 \otimes \sigma_1, & \Gamma_6 &= \mathbf{1} \otimes \sigma_2 \end{aligned} \quad (2.3)$$

where $\gamma_5 = \gamma^1 \gamma^2 \gamma^3 i \gamma^0$ and $\gamma_5^2 = \mathbf{1}$. In this representation the Weyl condition (2.2) becomes $\lambda = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$, with a complex four-component spinor ψ . The six-dimensional charge conjugation matrix satisfying $C_6 \Gamma_M C_6^{-1} = -\Gamma_M^T$ is $C_6 = C \gamma_5 \otimes \sigma_2$ where C is the four-dimensional charge conjugation matrix satisfying $C \gamma_\mu C^{-1} = -\gamma_\mu^T$. Since $C^T = -C$ and $(C \gamma_5)^T = -C \gamma_5$ one has $C_6^T = +C_6$.

The (3+1)-dimensional Lagrangian then reads

$$\begin{aligned} \mathcal{L} = & - \left\{ \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (D_\mu S)^2 + \frac{1}{2} (D_\mu P)^2 + \frac{1}{2} g^2 (S \times P)^2 \right\} \\ & - \left\{ \bar{\psi} \gamma^\mu D_\mu \psi + i g \bar{\psi} (S \times \psi) + g \bar{\psi} \gamma_5 (P \times \psi) \right\}. \end{aligned} \quad (2.4)$$

In the trivial sector we choose the symmetry-breaking Higgs field to be S^3 with background $S^a \equiv A_6^a = v \delta_3^a$; in the nontrivial sector we consider a static monopole background with

only A_j^a ($j = 1, 2, 3$) and S^a nonvanishing. The classical Hamiltonian density can be written in BPS form

$$\mathcal{H} = \frac{1}{4}(F_{ij}^a - \epsilon_{ijk}D_k S^a)^2 + \frac{1}{2}\partial_k(\epsilon_{ijk}F_{ij}^a S^a). \quad (2.5)$$

(One can also write \mathcal{H} in similar form if A_0 is nonvanishing (dyons) [3, 46, 42, 47], but we shall not need this extension.) Thus the classical BPS equations for the monopole read

$$F_{ij}^a - \epsilon_{ijk}D_k S^a = 0. \quad (2.6)$$

One can interpret this equation as a selfduality condition for F_{MN} with $M, N = 1, 2, 3, 6$ and $\epsilon_{ijk6} = \epsilon_{ijk}$. The anti-monopole is obtained by reversing the sign of S^a in eqs. (2.6) and (2.5).

To solve the BPS equation one may set $A_i^a = \epsilon_{aij}\frac{x^j}{gr^2}f(r)$ and $S^a = -m\frac{x^a}{gr}h(r)$ with $m = gv$ and impose the boundary conditions $f(r) \rightarrow 1$ and $h(r) \rightarrow 1$ for $r \rightarrow \infty$. This leads to two coupled first-order ordinary differential equations for $f(r)$ and $h(r)$,

$$-2f + f^2 + r^2mh' = 0, \quad f' = mh(1 - f), \quad (2.7)$$

whose solution was obtained by Prasad and Sommerfield [48]

$$f = 1 - mr/\sinh(mr), \quad h = \coth(mr) - (mr)^{-1}. \quad (2.8)$$

For us only the asymptotic values of the background fields will be needed. These are given by

$$\begin{aligned} A_i^a &\rightarrow \epsilon_{aij}\frac{1}{g}\frac{\hat{x}^j}{r}, & F_{ij}^a &\rightarrow -\epsilon_{ijk}\frac{1}{g}\frac{\hat{x}^a\hat{x}^k}{r^2}, \\ S^a &\rightarrow -\delta_i^a\hat{x}^i v \left(1 - \frac{1}{mr}\right), & D_i S^a &\rightarrow -\frac{1}{g}\frac{\hat{x}^i\hat{x}^a}{r^2}, \end{aligned} \quad (2.9)$$

where $\hat{x}^i \equiv x^i/r$. Substituting these expressions into the surface term in the Hamiltonian (2.5) yields the classical mass

$$M_{\text{cl.}} = 4\pi m/g^2. \quad (2.10)$$

Note that the Hamiltonian (2.5) corresponds to the standard expression for the gravitational stress tensor associated with the Lagrangian density in (2.1). As we shall discuss further below, one can also define an improved stress tensor, and then one obtains in fact a different value for the classical mass.

The action is invariant under susy transformations with a complex chiral spinor $\eta = \begin{pmatrix} \epsilon \\ 0 \end{pmatrix}$

$$\delta A_M^a = \bar{\lambda}^a \Gamma_M \eta - \bar{\eta} \Gamma_M \lambda^a, \quad \delta \lambda^a = \frac{1}{2} F_{MN}^a \Gamma^M \Gamma^N \eta, \quad \delta \bar{\lambda}^a = -\frac{1}{2} \bar{\eta} F_{MN}^a \Gamma^M \Gamma^N. \quad (2.11)$$

The susy current as obtained from the Noether method applied to transformations with $\bar{\eta}$ reads

$$j^M = \frac{1}{2} \Gamma^P \Gamma^Q F_{PQ}^a \Gamma^M \lambda^a \equiv \frac{1}{2} \Gamma^{PQ} F_{PQ}^a \Gamma^M \lambda^a. \quad (2.12)$$

It is gauge invariant and conserved on-shell. Its susy variation yields⁵

$$\begin{aligned} \frac{1}{2}\delta j^M &= \frac{1}{8}(\Gamma^{PQMRS}\eta)F_{PQ}F_{RS} + \Gamma^N\eta(F^{MP}F_{NP} - \frac{1}{4}\delta_N^M F^{RS}F_{RS}) \\ &\quad - ((D^N\bar{\lambda})\Gamma^M\lambda)\Gamma_N\eta + (\lambda^T C_6\Gamma^M D_N\lambda)\Gamma^N\tilde{\eta} \end{aligned} \quad (2.13)$$

where $\tilde{\eta} = C_6^{-1}\bar{\eta}^T$ and we have dropped terms proportional to the equation of motion $\Gamma^M D_M\lambda = 0$ because charges are Heisenberg operators, which are on-shell. The terms with $\Gamma^N\eta$ should give the stress tensor T_{MN} . For the fermions one finds $-(D_N\bar{\lambda})\Gamma_M\lambda$, which seems to differ from the gravitational stress tensor⁶ $-(\delta S/\delta e_A^M)e_{AN} = \frac{1}{2}\bar{\lambda}\Gamma_N\vec{D}_M\lambda$. However, on-shell the gravitational stress tensor is symmetric as a consequence of the local Lorentz invariance of the action in curved space, and $-(D_N\bar{\lambda})\Gamma_M\lambda$ differs from $\frac{1}{2}\bar{\lambda}\Gamma_N\vec{D}_M\lambda$ by the total derivative $-\frac{1}{2}\partial_N(\bar{\lambda}\Gamma_M\lambda)$. This total derivative is separately conserved with respect to the index M , and does not contribute to the translation generators, $\frac{1}{2}\int\partial_N(\bar{\lambda}\Gamma^0\lambda)d\vec{x} = 0$.⁷ Thus the susy algebra does not depend on which stress tensor one uses, and for the energy it does not matter whether one uses $\int T_{00} = -\int(D_0\bar{\lambda})\Gamma_0\lambda$ or $\int T_{00} = \frac{1}{2}\int(\bar{\lambda}\Gamma_0\vec{D}_0\lambda)$.

Integrating over 3-space, the term with $\tilde{\eta}$ vanishes⁷ since

$$\lambda^T C_6\Gamma^M D_N\lambda = \frac{1}{2}\partial_N(\lambda^T C_6\Gamma^M\lambda),$$

while the terms with η yield the $N = 2$ susy algebra

$$\frac{i}{2}\{Q^\alpha, \bar{Q}_\beta\} = (\gamma^\mu)^\alpha{}_\beta P_\mu - (\gamma_5)^\alpha{}_\beta \hat{U} + i\delta_\beta^\alpha \hat{V}. \quad (2.14)$$

Here $P_\mu = \int T^0{}_\mu d^3x$ and \hat{U} and \hat{V} are the two real (or one complex) central charges of the $N = 2$ algebra, given explicitly by⁸

$$\hat{U} = \int \left[\frac{1}{2}\epsilon^{ijk}\partial_k(F_{ij}^a S^a) + \partial_i(P^a F_{i0}^a) \right] d^3x, \quad (2.15)$$

$$\hat{V} = \int \left[\frac{1}{2}\epsilon^{ijk}\partial_k(F_{ij}^a P^a) - \partial_i(S^a F_{i0}^a) \right] d^3x. \quad (2.16)$$

For a monopole background $\hat{U} = \int \frac{1}{2}\epsilon^{ijk}\partial_k(F_{ij}^a S^a)d^3x$ which saturates the BPS bound classically, see (2.5). Performing the integration over 3-space with a monopole background yields $\hat{U}_{cl.} = 4\pi m/g^2$.

⁵We have displayed the result for $\frac{1}{2}\delta j^M$ instead of δj^M in order to obtain the stress tensor with unit normalization on the right-hand side.

⁶To obtain the gravitational stress tensor, one may replace $\Gamma^A\Gamma^{BC}$ (with A, B, C flat vector indices) in the term $-(\bar{\lambda}\Gamma^A\Gamma^{BC}\lambda)e_A^M\frac{1}{4}\omega_{MBC}$ in the action by one-half the commutator because the anticommutator only yields terms with two or more vielbein fluctuation fields $e_A^M - \delta_A^M$. One must then evaluate the variation of $ee_A^M\omega_{MB}^A$ which is easiest obtained by using the vielbein postulate $\partial_M(ee_B^M) + ee_A^M\omega_{MB}^A = 0$. This yields $\delta S = \int[-\bar{\lambda}\Gamma^A D_M^{(YM)}\lambda + \frac{1}{2}\partial_M(\bar{\lambda}\Gamma^A\lambda)]\delta(ee_A^M)d^4x$ up to terms quadratic in $e_A^M - \delta_A^M$, where $D_M^{(YM)}$ is the flat-space Yang-Mills covariant derivative.

⁷For $N = j$ this is clear, but it also holds for $N = 0$ if one uses the Dirac equation.

⁸We have added hats to \hat{U} and \hat{V} to distinguish them from the $N = 4$ case, where we shall introduce the four central charges $U, \tilde{U}, V, \tilde{V}$.

It is well-known that one can add improvement terms $\Delta T_{\mu\nu}^{\text{impr}}$ to the stress tensor for scalar fields so that the improved stress tensor becomes traceless on-shell

$$T_{\mu\nu}(S) + \Delta T_{\mu\nu}^{\text{impr}}(S) = D_\mu S D_\nu S - \frac{1}{2} \eta_{\mu\nu} (D_\rho S)^2 - \frac{1}{6} (\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^2) S^2 \quad (2.17)$$

and similarly for P . (One may verify that the complete classical stress tensor for A_μ, P, S, ψ is indeed traceless when the full nonlinear field equations for P and S are satisfied, even though some of the fields are massive due to the Higgs effect.)

This corresponds to the possibility of introducing improvement terms to the susy current so that the improved susy current satisfies $\gamma^\mu j_\mu = 0$ (“gamma-tracelessness”) in 3+1 dimensions. In 5+1-dimensional notation

$$j_{\text{impr}}^\mu \equiv j^\mu + \Delta j_{\text{impr}}^\mu = \frac{1}{2} \Gamma^{PQ} F_{PQ} \Gamma^\mu \lambda - \frac{2}{3} \Gamma^{\mu\nu} \partial_\nu (A_{\mathcal{J}} \Gamma^{\mathcal{J}} \lambda) \quad (2.18)$$

where the index \mathcal{J} runs over $\mathcal{J} = 5, 6$ in the $N = 2$ model (in the $N = 4$ model it will run from 5 to 10). In fact, the improved currents on the one hand, and the improvement terms on the other hand, form separate $N = 2$ multiplets. Notice that the improvement terms are perfectly gauge invariant quantities in 3+1 dimensions, but not in the higher-dimensional ancestor model because there $A_{\mathcal{J}}$ are components of a gauge field.

A susy variation of the improvement term by itself yields the following result

$$\begin{aligned} \frac{1}{2} \delta(\Delta j_{\text{impr}}^\mu) &= -\frac{1}{6} \gamma^{\mu\nu} \partial_\nu (A_{\mathcal{J}} \Gamma^{\mathcal{J}} F_{MN} \Gamma^{MN} \eta) \\ &\quad - \frac{1}{3} \gamma^{\mu\nu} \partial_\nu (\Gamma^{\mathcal{J}} \lambda \{ \bar{\lambda} \Gamma_{\mathcal{J}} \eta - \bar{\eta} \Gamma_{\mathcal{J}} \lambda \}). \end{aligned} \quad (2.19)$$

The bosonic terms in the $\mu = 0$ component of this expression yield

$$\begin{aligned} &-\frac{1}{3} \gamma^0 \gamma^j \partial_j (P \gamma_5 + iS) \left(\frac{1}{2} F_{\rho\sigma} \gamma^{\rho\sigma} + D_\rho P \gamma^\rho \gamma_5 + D_\rho S i \gamma^\rho \right) \eta \\ &= -\frac{1}{3} \gamma^0 \gamma^j \partial_j \left[(P \gamma_5 + iS) \frac{1}{2} F_{\rho\sigma} \gamma^{\rho\sigma} - \frac{1}{2} \not{D} (S^2 + P^2) - i \gamma^\rho \gamma_5 P \overleftrightarrow{D}_\rho S \right] \eta. \end{aligned} \quad (2.20)$$

The terms with $S^2 + P^2$ yield the improvement terms of the stress tensor, and the terms with $(\gamma_5)^{\alpha\beta}$ and $i\delta^{\alpha\beta}$ are equal to $-\frac{1}{3}$ times the results in (2.14), (2.15), and (2.16). There are, however, further terms with a different spinor structure and while the $N = 4$ case has been worked out at the linearized level in ref. [49, 50], the complete multiplet of $N = 2$ improvement terms seems not to be known. There are also fermionic terms, which read after a Fierz rearrangement

$$\frac{1}{12} \gamma^{\mu\nu} \partial_\nu [(\bar{\lambda} O^I \lambda) \Gamma_{\mathcal{J}} O_I \Gamma^{\mathcal{J}} \eta + (\lambda^T C_6 O^I \lambda) \Gamma_{\mathcal{J}} O_I \Gamma^{\mathcal{J}} \tilde{\eta}] \quad (2.21)$$

where we used $\bar{\eta} \Gamma_{\mathcal{J}} \lambda = -\lambda^T \Gamma_{\mathcal{J}}^T \bar{\eta}^T = (\lambda^T C_6) \Gamma_{\mathcal{J}} (C_6^{-1} \bar{\eta}^T) = \lambda^T C_6 \Gamma_{\mathcal{J}} \tilde{\eta}$. Since $\lambda^T C_6 O^I \lambda$ is only nonvanishing for $O^I = \Gamma^M, \Gamma^{M_1 \dots M_5}$, we find no fermionic improvement terms for the central charges \hat{U} and \hat{V} .

As we have seen, the improvement terms for the central charges \hat{U} and \hat{V} are proportional to the unimproved ones,

$$\Delta\hat{U}^{\text{impr}} = -\frac{1}{3}\hat{U}. \quad (2.22)$$

This shows that using an improved supercurrent multiplet reduces the classical value for the monopole mass and central charge to 2/3 of its conventional (unimproved) value. (For the mass this can be verified explicitly by evaluating the improvement term (2.17) using the asymptotic results for the scalar field of eq. (2.9).)

2.2 $N = 4$

We now turn to the $N = 4$ super Yang-Mills theory with monopoles. Most aspects are the same as in the $N = 2$ case, but for the improvement terms there are important differences.

The $N = 4$ model is most easily obtained by dimensional reduction from 9+1 dimensions [44, 51]. The Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4}F_{MN}^2 - \frac{1}{2}\bar{\lambda}\Gamma^M D_M\lambda \\ &= -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}(D_\mu S_j)^2 - \frac{1}{2}(D_\mu P_j)^2 - \frac{1}{2}\bar{\lambda}^I \not{D}\lambda^I + \text{interactions} \end{aligned} \quad (2.23)$$

where the factor $\frac{1}{2}$ is due to the fact that the fermionic field λ is now a Majorana-Weyl spinor, and the indices M, N now run over $0, \dots, 3, 5, \dots, 10$. After dimensional reduction there are three adjoint scalar and pseudoscalar fields, indexed by $j = 1, 2, 3$, and four adjoint Majorana fields indexed by $I = 1, \dots, 4$, but it will be often convenient to keep the 10-dimensional notation. The action is invariant under $\delta A_M = -\bar{\epsilon}\Gamma_M\lambda = \bar{\lambda}\Gamma_M\epsilon$ and $\delta\lambda = \frac{1}{2}F_{MN}\Gamma^{MN}\epsilon$.

The susy current is again formally given by (2.12) but now varies under susy into

$$\begin{aligned} \frac{1}{2}\delta j^M(x) &= \left(F^{MP}F_{NP} - \frac{1}{4}\delta_N^M F^{RS}F_{RS} + \frac{1}{2}\bar{\lambda}\Gamma^M D_N\lambda \right) \Gamma^N\epsilon \\ &\quad + \frac{1}{16}(\bar{\lambda}\Gamma^M\Gamma^{PQ}D^R\lambda)\Gamma_{PQR}\epsilon \\ &\quad + \frac{1}{8}F_{NP}F_{QR}\Gamma^{NPMQR}\epsilon \end{aligned} \quad (2.24)$$

The term with Γ_{PQR} vanishes after integration over x .⁹

For the purpose of dimensional reduction, the 16-component Majorana-Weyl spinor λ^a is written as $(\lambda^{\alpha I a}, 0)$, where $\alpha = 1, \dots, 4$ is the 4-dimensional spinor index, $I = 1, \dots, 4$ is the rigid SU(4) index, and a is the adjoint SU(2) colour index. For the gamma matrices we use the representation¹⁰

$$\Gamma_\mu = \gamma_\mu \otimes \mathbf{1} \otimes \sigma_2, \quad \mu = 0, 1, 2, 3,$$

⁹One can rewrite $(\bar{\lambda}\Gamma^M\Gamma^{PQ}D^R\lambda)\Gamma_{PQR}\epsilon$ as $\bar{\lambda}\{\frac{1}{6}[\Gamma^{MPQR}, \Gamma^N]D_N\lambda + \frac{1}{3}\bar{\lambda}\Gamma^{PQR}D^M\lambda + \frac{1}{2}\bar{\lambda}[\Gamma^{QR}, \Gamma^N]D_N\lambda\}\Gamma^M_{QR}\epsilon$ and $\int \bar{\lambda}\Gamma^{PQR}D^M\lambda$ vanishes since $\bar{\lambda}\Gamma^{PQR}D^M\lambda = (D^M\bar{\lambda})\Gamma^{PQR}\lambda$ is a total derivative of fermionic terms.

¹⁰Majorana spinors satisfy $\lambda^T C_{10} = \lambda^\dagger i\Gamma^0$ in 9+1 dimensions and $(\lambda^I)^T C_4 = (\lambda^I)^\dagger i\gamma^0$ in 3+1 dimensions.

$$\begin{aligned}
\Gamma_{4+j} &= \mathbf{1} \otimes \alpha_j \otimes \sigma_1, \\
\Gamma_{7+j} &= \gamma_5 \otimes \beta_j \otimes \sigma_2, \quad j = 1, 2, 3, \\
\Gamma_{11} &= \mathbf{1} \otimes \mathbf{1} \otimes \sigma_3, \quad C_{10} = -iC_4 \otimes \mathbf{1} \otimes \sigma_1.
\end{aligned} \tag{2.25}$$

The α_j and β_j are the six generators of $\text{SO}(4) = \text{SO}(3) \times \text{SO}(3)$ in the representation of purely imaginary antisymmetric 4×4 matrices [52, 53], self-dual and anti-self-dual, respectively, and satisfying $\{\alpha_i, \alpha_j\} = \{\beta_i, \beta_j\} = 2\delta_{ij}$, $[\alpha_i, \alpha_j] = 2i\epsilon_{ijk}\alpha_k$, $[\beta_i, \beta_j] = 2i\epsilon_{ijk}\beta_k$, and $[\alpha_j, \beta_k] = 0$. An explicit representation is given by

$$\begin{aligned}
\alpha_1 &= \sigma_1 \otimes \sigma_2, & \alpha_2 &= -\sigma_3 \otimes \sigma_2, & \alpha_3 &= \sigma_2 \otimes \mathbf{1} \\
\beta_1 &= \sigma_2 \otimes \sigma_1, & \beta_2 &= \mathbf{1} \otimes \sigma_2, & \beta_3 &= \sigma_2 \otimes \sigma_3.
\end{aligned} \tag{2.26}$$

Upon reduction to 3+1 dimensions, the spin zero fields $A_{\mathcal{J}}$ with $\mathcal{J} = 5, \dots, 10$ split into three scalars $S_j = A_{4+j}$ and three pseudoscalars $P_j = A_{7+j}$.

The 12 real central charges appear as¹¹

$$\begin{aligned}
\frac{1}{2}\{Q^{\alpha I}, Q^{\beta J}\} &= \delta^{IJ}(\gamma^\mu C^{-1})^{\alpha\beta} P_\mu \\
&+ i(\gamma_5 C^{-1})^{\alpha\beta} (\alpha^j)^{IJ} \int d^3x U_j - (C^{-1})^{\alpha\beta} (\beta^j)^{IJ} \int d^3x V_j \\
&+ (C^{-1})^{\alpha\beta} (\alpha^j)^{IJ} \int d^3x \tilde{V}_j + i(\gamma_5 C^{-1})^{\alpha\beta} (\beta^j)^{IJ} \int d^3x \tilde{U}_j,
\end{aligned} \tag{2.27}$$

where U_j and V_j are due to the five-gamma term in (2.24), and \tilde{U}_j and \tilde{V}_j due to the one-gamma terms. The indices I and J are lowered and raised by the charge conjugation matrix in this space, which is δ^{IJ} , see (2.25).

The $N = 2$ monopole (and dyon) can be embedded into the $N = 4$ model by selecting e.g. $j = 1$ and one then has ($S = S_1, P = P_1, U = U_1$ etc.)¹²

$$\begin{aligned}
U &= \partial_i \left(S^a \frac{1}{2} \epsilon^{ijk} F_{jk}^a \right), & \tilde{U} &= \partial_i (P^a F_{0i}^a) \\
V &= \partial_i \left(P^a \frac{1}{2} \epsilon^{ijk} F_{jk}^a \right), & \tilde{V} &= \partial_i (S^a F_{0i}^a).
\end{aligned} \tag{2.28}$$

The monopole background fields are again given by (2.9).

Let us now discuss the improvement terms. For the stress tensor they are given by

$$\Delta T_{\mu\nu}^{\text{impr}} = -\frac{1}{6}(\partial_\mu \partial_\nu - \eta_{\mu\nu} \partial^2)(A_{\mathcal{J}} A^{\mathcal{J}}) \tag{2.29}$$

¹¹The complex matrix $Z^{IJ} = -Z^{JI}$ of central charges contains 6 complex (12 real) elements, as shown in (2.27). The magnetic charge U_1 and the electric charge \tilde{V}_1 only appear in the combination $Z^{IJ} = (iU_1 + \tilde{V}_1)(\alpha^1)^{IJ} + \dots$ in the left-handed sector. By a unitary transformation one can block-diagonalize Z^{IJ} , with two real antisymmetric 2×2 matrices along the diagonal. The $N = 4$ action has a rigid R symmetry group $\text{SU}(4)$ (not $\text{U}(4)$ [45]). To exhibit this $\text{SU}(4)$, the spin zero fields are combined into $M^{IJ} = (\alpha^j)^{IJ} S_j + i(\beta^j)^{IJ} P_j$, but if $S_1^a = v\delta_3^a$, there is a central charge acting on excitations and R is broken down to the manifest stability subgroup of $\langle M^{23} \rangle = \langle M^{14} \rangle = S_1$, which is $\text{USp}(4)$. (Note that the central charge and all susy generators vanish on the trivial spontaneously broken vacuum).

¹²To obtain the total derivatives in (2.28), one needs to use Bianchi identities in the case of U and V , and equations of motion in the case of \tilde{U} and \tilde{V} .

where $\mathcal{J} = 5, \dots, 10$. Also there is an improvement term in the supercurrent

$$\frac{1}{2}\Delta j_{\text{impr}}^\mu = -\frac{1}{3}\Gamma^{\mu\nu}\partial_\nu(A_{\mathcal{J}}\Gamma^{\mathcal{J}}\lambda) \quad (2.30)$$

The susy variation of $\Delta j_{\text{impr}}^\mu$ is simpler to evaluate than in the $N = 2$ case because both λ and ϵ are Majorana spinors. One finds

$$\frac{1}{2}\delta(\Delta j_{\text{impr}}^\mu) = -\frac{1}{3}\Gamma^{\mu\nu}\partial_\nu \left[A_{\mathcal{J}}\Gamma^{\mathcal{J}}\frac{1}{2}F_{MN}\Gamma^{MN}\epsilon - (\bar{\epsilon}\Gamma_{\mathcal{J}}\lambda)\Gamma^{\mathcal{J}}\lambda \right] \quad (2.31)$$

The bosonic terms read more explicitly

$$-\frac{1}{6}\gamma^{\mu\nu}\partial_\nu(\alpha^j \otimes \sigma_1 S_j + \gamma_5 \otimes \beta^j \otimes \sigma_2 P_j)(\gamma^{\rho\sigma} F_{\rho\sigma} - 2i\gamma^\rho \alpha^l \otimes \sigma_3 D_\rho S_l + 2\gamma^\rho \gamma_5 \beta^l D_\rho P_l)\epsilon. \quad (2.32)$$

They yield again $-\frac{1}{3}$ times the unimproved central charges. From the fermionic terms one obtains for $\mu = 0$ after a Fierz rearrangement

$$\frac{1}{3}\Gamma^{0j}\partial_j \left[\frac{1}{24}(\bar{\lambda}\Gamma^{PQR}\lambda) \right] \Gamma_{PQR}\epsilon = -\frac{1}{3}\epsilon^{jkl}\partial_j \left[\frac{i}{8}\bar{\lambda}\gamma_{kl}\alpha^1\lambda \right] (i\gamma_5\alpha^1\epsilon) + \dots \quad (2.33)$$

Thus one obtains the following improvement terms for the central charges

$$\Delta U_1^{\text{impr}} = -\frac{1}{3} \left[U_1 + \int \frac{i}{8}\partial_i(\epsilon^{ijk}\bar{\lambda}\alpha^1\gamma_{jk}\lambda)d^3x \right] \quad (2.34)$$

etc. Classically, there is no contribution from the fermionic terms, so like in the $N = 2$ the use of improved susy currents and stress tensors leads to a mass and central charge of a monopole equal to $2/3$ of their conventional (unimproved) value.

3. Fluctuations about the monopole background and renormalization

To perform quantum calculations one needs to add a gauge fixing term and ghosts to the classical Lagrangian. Usually fermions do not enter the gauge fixing condition. Therefore, as for the bosonic Lagrangian, we formally don't have to differentiate between the $N = 2$ and the $N = 4$ model in this respect. The background-covariant Feynman- R_ξ gauge at $\xi = 1$, implemented in the higher dimensional space, has turned out to be the most convenient gauge in the quantum theory of solitons [39, 35]. For general ξ the BRS-exact gauge fixing Lagrangian including ghosts is given by¹³

$$\mathcal{L}_{\text{gf+gh}} = -\frac{1}{2\xi}[D_M(A)a^M]^2 - [D^M(A)b] D_M(A+a)c, \quad (3.1)$$

where a_M are the quantum fields and A_M the background fields.

¹³Note that for $\xi \neq 1$ the kinetic terms of the quantum fluctuations are not diagonal; for that one would need the non-background covariant gauge of the form $-\frac{1}{2\xi}[\partial_\mu a^\mu + \xi(\dots)]^2$.

3.1 Fluctuation equations and propagators

For the calculation of one-loop corrections we need the terms in the action that are quadratic in quantum fluctuations about a (monopole) background. Propagators and the fluctuation equations are then obtained from this quadratic Lagrangian. Expanding the bosonic Lagrangian in (2.4) and the gauge fixing Lagrangian (3.1) around the solution of (2.6), $A_\mu \rightarrow A_\mu + a_\mu$, $S \rightarrow S + s$, $P \rightarrow p$, the quadratic Lagrangian reads for $\xi = 1$ and $N = 2$

$$\begin{aligned} \mathcal{L}_{\text{bos+gh}}^{(2)} = & -\frac{1}{2} a_{\underline{m}} \left[(\partial_0^2 - \partial_5^2 - D_{\underline{\ell}}^2) \delta_{\underline{mn}} - 2gF_{\underline{mn}} \times \right] a_{\underline{n}} \\ & -\frac{1}{2} p (\partial_0^2 - \partial_5^2 - D_{\underline{m}}^2) p \\ & +\frac{1}{2} a_0 (\partial_0^2 - \partial_5^2 - D_{\underline{m}}^2) a_0 - b (\partial_0^2 - \partial_5^2 - D_{\underline{m}}^2) c, \end{aligned} \quad (3.2)$$

where we kept the derivative w.r.t. one of the extra dimensions, ∂_5 , for the purpose of regularization. The quantum fields a_i and s are combined in a quartet $a_{\underline{m}} = (a_i, s)$. The background field strength $F_{\underline{mn}}$ and the background covariant derivative $D_{\underline{m}}$ are defined correspondingly.¹⁴ The $N = 4$ model has four additional (pseudo)scalars q_I with the same quadratic Lagrangian as for the pseudoscalar p .

The linearized field equations, i.e the fluctuation equations, are given by

$$\begin{aligned} \left[(\partial_0^2 - \partial_5^2 - D_{\underline{\ell}}^2) \delta_{\underline{mn}} - 2gF_{\underline{mn}} \times \right] a_{\underline{n}} &= 0, \\ (\partial_0^2 - \partial_5^2 - D_{\underline{m}}^2) (p, a_0, b, c) &= 0. \end{aligned} \quad (3.3)$$

The propagators are given correspondingly by

$$\langle a_{\underline{m}}^b(x) a_{\underline{n}}^c(y) \rangle = i \langle x | [(\partial_5^2 - \partial_0^2) \delta^{bc} + (D_{\underline{k}}^2)^{bc} \delta_{\underline{mn}} + 2g\epsilon^{bac} F_{\underline{mn}}^a]^{-1} | y \rangle, \quad (3.4)$$

$$\langle p^b(x) p^c(y) \rangle = i \langle x | [(\partial_5^2 - \partial_0^2) \delta^{bc} + (D_{\underline{k}}^2)^{bc}]^{-1} | y \rangle. \quad (3.5)$$

Note that the propagators for the ghosts and the component a_0 have an opposite sign, i.e.

$$\langle p^b(x) p^c(y) \rangle = -\langle a_0^b(x) a_0^c(y) \rangle = -\langle b^b(x) c^c(y) \rangle. \quad (3.6)$$

The fermionic fluctuations are independent of the gauge condition chosen for the bosonic fields. For the $N = 2$ model they are obtained from the dimensional reduction of $\Gamma^M D_M \lambda = 0$, which using (2.3) gives

$$(\gamma^\mu D_\mu + \gamma_5 \partial_5) \psi + igS \times \psi = 0. \quad (3.7)$$

In order to obtain a diagonal action for the two complex 2-component spinors into which ψ of the $N = 2$ model decomposes,¹⁵ $\psi = \begin{pmatrix} \psi_+ \\ i\psi_- \end{pmatrix}$, we use an unconventional representation for the 4-dimensional Dirac matrices,

$$\gamma^k = \begin{pmatrix} \sigma^k & 0 \\ 0 & -\sigma^k \end{pmatrix}, \quad \gamma^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (3.8)$$

¹⁴The various boundary terms that we omitted in (3.2) may give contributions at the boundary of space-time but they are not included in the definition of the propagators.

¹⁵The factor i in front of ψ_- is inserted in order that the field operators for ψ_+ and ψ_- have the same form as for the kink and the vortex that we shall discuss in section 5.

One finds then the following two field equations for ψ_+ and ψ_- in a monopole background

$$\mathcal{D}\psi_+ = (\partial_0 - \partial_5)\psi_-, \quad \bar{\mathcal{D}}\psi_- = (\partial_0 + \partial_5)\psi_+, \quad (3.9)$$

where the operators in (3.9) are defined by

$$\mathcal{D} := \sigma^{\underline{m}} D_{\underline{m}} = \sigma^k D_k + igS \times, \quad \bar{\mathcal{D}} := \bar{\sigma}^{\underline{m}} D_{\underline{m}} = \sigma^k D_k - igS \times. \quad (3.10)$$

Here we have introduced the Euclidean sigma matrices $\sigma^{\underline{m}} = (\sigma^k, i\mathbf{1})$ and $\bar{\sigma}^{\underline{m}} = (\sigma^k, -i\mathbf{1})$. Iteration yields

$$\bar{\mathcal{D}}\mathcal{D}\psi_+ = (\partial_0^2 - \partial_5^2)\psi_+, \quad \mathcal{D}\bar{\mathcal{D}}\psi_- = (\partial_0^2 - \partial_5^2)\psi_-. \quad (3.11)$$

In terms of the self-dual background $F_{\underline{mn}}$ one obtains

$$\bar{\mathcal{D}}\mathcal{D} = D_{\underline{m}}^2 + \frac{1}{2} \bar{\sigma}^{\underline{mn}} g F_{\underline{mn}}, \quad \mathcal{D}\bar{\mathcal{D}} = D_{\underline{m}}^2, \quad (3.12)$$

where we have used the self-dual Lorentz generators $\bar{\sigma}^{\underline{mn}} = \frac{1}{2}(\bar{\sigma}^m \sigma^n - \bar{\sigma}^n \sigma^m)$.

Introducing spinor notation for the quartet $\underline{a}_{\underline{m}} = (a_i, s)$, i.e. $\not{a} = \sigma^{\underline{m}} \underline{a}_{\underline{m}}$ and $\bar{\not{a}} = \bar{\sigma}^{\underline{m}} \underline{a}_{\underline{m}}$, one can rewrite the bosonic part of the Lagrangian quadratic in quantum fields in spinor notation as follows

$$\begin{aligned} \mathcal{L}_{\text{bos.}}^{(2)} = & \frac{1}{4} \text{tr} \left[\not{a} (\bar{\mathcal{D}}\mathcal{D} + \partial_5^2 - \partial_0^2) \not{a} - a_0 (\mathcal{D}\bar{\mathcal{D}} + \partial_5^2 - \partial_0^2) a_0 \right. \\ & \left. + p (\mathcal{D}\bar{\mathcal{D}} + \partial_5^2 - \partial_0^2) p + 2b (\mathcal{D}\bar{\mathcal{D}} + \partial_5^2 - \partial_0^2) c \right]. \end{aligned} \quad (3.13)$$

Thus the four bosonic fields \not{a} satisfy the same fluctuation equations as ψ_+ , whereas the quartet¹⁶ (a_0, p, b, c) satisfies the same equations as ψ_- . In the $N = 4$ case the additional scalars q_I of that theory have the same fluctuation equations as this quartet.

Although the gauge fixing term in (3.1) is not susy invariant, for $\xi = 1$ it gives rise to a (quantum mechanically) supersymmetric set of fluctuation equations. The factorization property (3.12) implies that the two operators $\mathcal{D}\bar{\mathcal{D}}$ and $\bar{\mathcal{D}}\mathcal{D}$ are isospectral, except for zero modes. The respective normalized eigen-modes of ψ_+ and ψ_- with mode energy $\omega_k^2 = k^2 + m^2$,

$$-\bar{\mathcal{D}}\mathcal{D}\chi_k^+ = \omega_k^2 \chi_k^+, \quad -\mathcal{D}\bar{\mathcal{D}}\chi_k^- = \omega_k^2 \chi_k^-, \quad (3.14)$$

are related to each other as follows:

$$\chi_k^- = \frac{1}{\omega_k} \mathcal{D}\chi_k^+. \quad (3.15)$$

This susy quantum mechanical structure will allow us to use index theorems for the calculation of the difference of the spectral densities associated with the operators $\mathcal{D}\bar{\mathcal{D}}$ and $\bar{\mathcal{D}}\mathcal{D}$.

¹⁶This quartet is different from the usual Kugo-Ojima quartet [54]. The pseudo Goldstone fields are given by $s_1^{1,2}$. Note also that these are massive, whereas the Higgs field s_1^3 is massless.

In $4 + \epsilon$ dimensions the fermionic quantum field has the following mode expansion

$$\psi(x) = \begin{pmatrix} \psi_+ \\ i\psi_- \end{pmatrix} = \int \frac{d^\epsilon \ell}{(2\pi)^{\epsilon/2}} \not{\sum} \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} \left\{ b_{kl} e^{-i(\omega t - \ell x^5)} \begin{pmatrix} \sqrt{\omega + \ell} \chi_k^+ \\ -\sqrt{\omega - \ell} \chi_k^- \end{pmatrix} \right. \\ \left. + d_{kl}^\dagger e^{i(\omega t - \ell x^5)} \begin{pmatrix} \sqrt{\omega + \ell} \chi_k^+ \\ \sqrt{\omega - \ell} \chi_k^- \end{pmatrix} \right\} + \text{zero modes}, \quad (3.16)$$

where χ_k^\pm depend on x^1, x^2, x^3 and the regularized mode energies are given by

$$\omega^2 = \omega_k^2 + \ell^2. \quad (3.17)$$

The bosonic fields a_m in spinor notation and the quartet a_0, p, b, c have analogous mode expansions in terms of χ_k^- and χ_k^+ , respectively. For example

$$a_0(x) = \int \frac{d^\epsilon \ell}{(2\pi)^{\epsilon/2}} \not{\sum} \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega}} \left(a_{k\ell} e^{-i(\omega t - \ell x^5)} \chi_k^- + \text{h. c.} \right). \quad (3.18)$$

For the $N = 4$ model the dimensional reduction using (2.25) gives the following fluctuation equations for the fermions in the monopole background

$$\gamma^\mu D_\mu \psi_I - igS \times \alpha_{IJ}^1 \psi_J = 0, \quad (3.19)$$

where α_{IJ}^1 was given in (2.26) and we have omitted the regulating extra dimension since the $N = 4$ model is non-anomalous. The ψ_I are four 4-component $D = 3 + 1$ Majorana spinors. The mode expansions of these Majorana spinors contains an extra factor $1/\sqrt{2}$ compared to (3.16). One can decouple the equation in the $SU(4)$ -space by introducing complex linear combinations $\psi_{(I,II)} = \frac{1}{\sqrt{2}}(\psi_1 \pm i\psi_4)$ and $\psi_{(III,IV)} = \frac{1}{\sqrt{2}}(\psi_2 \pm i\psi_3)$. The resulting equations are

$$\begin{aligned} \gamma^\mu D_\mu \psi_{(I,III)} + igS \times \psi_{(I,III)} &= 0, \\ \gamma^\mu D_\mu \psi_{(II,IV)} - igS \times \psi_{(II,IV)} &= 0. \end{aligned} \quad (3.20)$$

The first two equations coincide with the $N = 2$ equation (3.7). (Setting $\psi_1 = i\psi_4$ and $\psi_2 = i\psi_3$, together with $\epsilon_1 = i\epsilon_4$ and $\epsilon_2 = i\epsilon_3$ as well as $P_2 = P_3 = S_2 = S_3 = 0$, is a consistent truncation from $N = 4$ to $N = 2$.)

3.2 One-loop counterterms

To calculate quantum corrections to solitons at one-loop order in a well-defined manner, we have to set up renormalization conditions and determine counterterms to the parameters in the Lagrangian. This is done by identifying the parameters and fields in the Lagrangian as bare quantities $g_0, A_{(0)\mu}^a, a_{(0)\mu}^a, \lambda_{(0)}, \dots$, and introducing renormalization constants according to $g = Z_g g_0, A_{(0)\mu}^a = \sqrt{Z_A} A_\mu^a, a_{(0)\mu}^a = \sqrt{Z_a} a_\mu^a, \lambda_{(0)} = \sqrt{Z_\lambda} \lambda, \dots$. For this it is sufficient to consider the spontaneously broken phase in the trivial background with no monopoles. Choosing $\langle S_1^a \rangle = v \delta_3^a$ and expanding the Lagrangian with R_ξ gauge fixing term (3.1) and $\xi = 1$ one finds that all fields with color index $a = 3$ are massless, whereas $a = 1, 2$ have mass $m = gv$.

Because our gauge-fixing term is background covariant, we have different vertices for bosonic background fields and bosonic quantum fields. Considering firstly tadpole diagrams with an external Higgs boson field, we find that they vanish for an external background field S_1 as well as for an external quantum field s_1 . This is fortunate because in our model there is no counterterm linear in s_1 or S_1 so that a nonvanishing tadpole divergence could not have been removed by renormalization.

For determining wave-function and coupling constant renormalization we shall only consider external background fields. The background covariance of (3.1) then fixes the coupling constant renormalization through wave-function renormalization of the gauge boson background fields according to $Z_g = Z_A^{-1/2}$. Using the two-point self-energy of the massless gauge boson with external background fields we require that Z_A absorbs the entire one-loop correction on mass-shell, which because of Lorentz invariance is simply obtained at vanishing external momentum. Because the massless gauge boson only couples to massive fields, this does not involve infrared divergences, and one finds in dimensional regularization after a Wick rotation

$$Z_A = Z_g^{-2} = 1 + 2(4 - N)g^2 I, \quad (3.21)$$

$$I \equiv \int \frac{d^{4+\epsilon} k}{(2\pi)^{4+\epsilon}} \frac{-i}{(k^2 + m^2)^2} = \int \frac{d^{4+\epsilon} k_E}{(2\pi)^{4+\epsilon}} \frac{1}{(k_E^2 + m^2)^2},$$

where $N = 2$ or 4 , and k_E are Euclidean 4-momenta. Note that in the case $N = 4$ the coupling constant and the (background) gauge field wave function do not renormalize, whereas for $N = 2$ there is infinite ultraviolet renormalization with $I^{\text{div}} = -\frac{1}{8\pi^2} \frac{1}{\epsilon}$.

Renormalizing similarly the massless Higgs boson on-shell, one finds $Z_S = Z_A$ in the gauge $\xi = 1$, but

$$Z_S = 1 + 2(4 - N + 1 - \xi)g^2 I \quad (3.22)$$

for $\xi \neq 1$. So for $\xi \neq 1$ there is a need for infinite scalar wave function renormalization also in $N = 4$. (Divergent wave-function renormalizations in $N = 4$ were previously obtained in ref. [55] for non-background-covariant gauges.)

The renormalization of the mass $m = gv$ of the fields with SU(2) index $a = 1, 2$ is fixed by $Z_v = Z_S$ and $m = Z_m m_0 = Z_g Z_S^{1/2} g_0 v_0$ so that for $\xi = 1$ the mass does not renormalize. However, for general ξ one has $Z_m = 1 + g^2(1 - \xi)I$ as we have verified by direct calculation (contradicting a different assertion in ref. [37]). This ξ dependence of Z_m is actually needed in order that the physical mass (the value at the pole) be ξ independent.

For the wave function renormalization of the fermions we do not have to distinguish between quantum and background fields. However, while the gauge fixing term (3.1) respects background gauge covariance, it is not susy invariant, so the wave function renormalization constant $\sqrt{Z_\lambda}$ of the fermions is not related to Z_A or Z_S , even when $\xi = 1$. A straightforward calculation of the one-loop fermion self-energy, renormalized such that it vanishes on-shell for the massless fermions, leads to

$$\lambda_0 = \sqrt{Z_\lambda} \lambda, \quad Z_\lambda = 1 - 2(N + \xi - 1)g^2 I \quad (3.23)$$

in N -extended super-Yang-Mills theory. This is divergent even for $\xi = 1$ and $N = 4$.

In the following we shall restrict ourselves to the case $\xi = 1$ as this is where we can employ index theorems to determine spectral densities and thus perform explicit calculations in the nontrivial (monopole) sector.

When considering composite operators such as the susy current and its susy variations, one has to allow for composite-operator renormalization. This is generally not simple multiplicative renormalization, but involves matrices of renormalization constants for a whole set of composite operators.

In ref. [56] it has been shown that the improved susy current $j_{\text{impr}}^\mu = j^\mu + \Delta j_{\text{impr}}^\mu$ is a finite operator and thus protected from renormalization. However, in ref. [40] we have found the somewhat surprising result that while in the case of $N = 2$ both the unimproved and the improved current are finite operators, not requiring renormalization beyond standard wave function and coupling constant renormalization, the situation is different in the $N = 4$ theory.

By considering explicitly the renormalization of the central charge operator U in eq. (2.28), we have found that for $N = 4$ it requires nonmultiplicative renormalization through improvement terms, similarly to what has been observed in nonsupersymmetric theories in the renormalization of the energy-momentum tensor [57–61]. However, because the improved susy current does not renormalize, it is only the improvement terms that acquire composite-operator renormalization, and it turns out that one can renormalize the latter multiplicatively. In ref. [40] we have obtained by explicit calculation

$$\begin{aligned} Z_{\Delta U_{\text{impr}}}^{N=2} &= 1, \\ Z_{\Delta U_{\text{impr}}}^{N=4} &= 1 + 12g^2 I, \end{aligned} \tag{3.24}$$

where we have renormalized again with external massless legs taken at vanishing momenta (which involves only massive loop integrals). This result is in fact ξ -independent [40]. Because the susy violation that is implied by the gauge fixing term (3.1) and the corresponding ghost Lagrangian is only in the form of BRS-exact terms, the entire susy multiplet of improvement terms has the same Z factor,

$$Z_{\Delta j_{\text{impr}}^\mu} = Z_{\Delta T_{\text{impr}}^{\mu\nu}} = Z_{\Delta U_{\text{impr}}}. \tag{3.25}$$

A consequence of this multiplicative renormalization of improvement terms in the $N = 4$ susy current is that the unimproved susy current $j^\mu = j_{\text{impr}}^\mu - \Delta j_{\text{impr}}^\mu$ receives an *additive* renormalization with counterterm

$$\delta j^\mu = -(Z_{\Delta U} - 1)\Delta j_{\text{impr}}^\mu. \tag{3.26}$$

4. Bulk and surface contributions to the quantum mass of susy monopoles

In this section we decompose the space integral of the Hamiltonian density for monopoles into surface terms and bulk terms. To one-loop order, the latter correspond to the usual sum over zero-point energies, but the former lead to new effects which are not present in lower-dimensional solitons and which we evaluate in section 6.2. We always begin with the $N = 2$ case, and then give the corresponding results for the $N = 4$ case.

In order to determine the energy-momentum tensor, we consider the quantum action in curved space. For the gauge fixed theory and $\xi = 1$ it reads, for the $N = 2$ model

$$\begin{aligned}
 S = \int d^4x \left\{ -\frac{1}{4} \sqrt{-g} g^{MR} g^{NS} F_{MN}(A+a) F_{RS}(A+a) \right. \\
 - \sqrt{-g} \bar{\lambda} \Gamma^M D_M(A+a) \lambda - \frac{1}{2} \frac{1}{\sqrt{-g}} [D_M(A) \sqrt{-g} g^{MN} a_N]^2 \\
 \left. - [D_M(A) b] \sqrt{-g} g^{MN} D_N(A+a) c \right\}. \tag{4.1}
 \end{aligned}$$

In the $N = 4$ case λ is Majorana and an extra factor $\frac{1}{2}$ has to be included in the fermionic term.

The matrix Γ^M is defined as the product of the constant matrices in (2.25) and a vielbein field. The stress tensor is defined by

$$T_{MN} = -2 \frac{\delta}{\delta g^{MN}} S \tag{4.2}$$

(and a similar formula with the vielbein field for spinors). We thus obtain, for $N = 2$ and on-shell (see also footnote 6),

$$\begin{aligned}
 T_{MN} = F_{MS} F_N^S - \frac{1}{4} \eta_{MN} F_{RS} F^{RS} + \frac{1}{2} \bar{\lambda} \Gamma_N \overleftrightarrow{D}_M \lambda \\
 - 2 a_{(M} \hat{D}_{N)} \hat{D}^R a^R + \eta_{MN} \left[a^S \hat{D}_S \hat{D}^R a^R + \frac{1}{2} (\hat{D}^R a^R)^2 \right] \\
 + 2 [\hat{D}_{(M} b] D_{N)} c - \eta_{MN} [\hat{D}^R b] D_{Rc}, \tag{4.3}
 \end{aligned}$$

where F_{MN} and D_M involve the full field $A+a$ and where we have temporarily introduced the notation $\hat{D}_M = D_M(A)$ for the background covariant derivative. For $N = 4$ the fermionic part of the stress tensor is instead $T_{MN}^{\text{ferm}} = \frac{1}{2} \bar{\lambda} \Gamma_N D_M \lambda$.

For the Hamiltonian density T_{00} the terms quadratic in quantum fields are given by

$$\begin{aligned}
 T_{00}^{(2)} = \left[\frac{1}{2} (F_{0\mathbf{S}})^2 + \frac{1}{4} F_{\mathbf{RS}}^2 \right]^{(2)} - 2a_0 D_0 D_R a^R - a^S D_S D_R a^R - \frac{1}{2} (D_R a^R)^2 \\
 + 2(D_0 b)(D_0 c) + (D^R b)(D_R c) + T_{00}^{(2)\text{ferm}} \tag{4.4}
 \end{aligned}$$

where the indices \mathbf{R}, \mathbf{S} run over the nonzero values of the indices R, S . We have dropped the hat on D as from here on the covariant derivatives involve only the background fields. The fermionic contribution is given by

$$T_{00}^{(2)\text{ferm}} = \begin{cases} \frac{1}{2} \bar{\lambda} \Gamma_0 \overleftrightarrow{D}_0 \lambda & \text{for } N = 2, \\ \frac{1}{2} \bar{\lambda} \Gamma_0 D_0 \lambda & \text{for } N = 4. \end{cases} \tag{4.5}$$

The bosonic contributions have the same formal structure in $N = 2$ and $N = 4$, only the range of the indices is different. The first two terms in (4.4) are the classical contribution to $T_{00}^{(2)}$ from the bosons. Expanding

$$F_{0\mathbf{S}}(A+a) = F_{0\mathbf{S}}(A) + D_0(A) a_{\mathbf{S}} - D_{\mathbf{S}}(A) a_0 + g a_0 \times a_{\mathbf{S}} \tag{4.6}$$

we obtain

$$\begin{aligned} \left[\frac{1}{2} (F_{0\mathbf{S}})^2 \right]^{(2)} &= -\frac{1}{2} a_{\mathbf{S}} D_0^2 a_{\mathbf{S}} - \frac{1}{2} a_0 D_{\mathbf{S}}^2 a_0 + a_{\mathbf{S}} D_{\mathbf{S}} D_0 a_0 + 2g F_{0\mathbf{S}} a_0 \times a_{\mathbf{S}} \\ &\quad + \frac{1}{2} \partial_0 (a_{\mathbf{S}} D_0 a_{\mathbf{S}}) + \frac{1}{2} \partial_s (a_0 D_s a_0) - \partial_0 (a_{\mathbf{S}} D_{\mathbf{S}} a_0), \end{aligned} \quad (4.7)$$

where we wrote all terms involving two derivatives in the form $aDDa$ (“bulk terms”) plus total derivatives, and used $a_{\mathbf{S}} D_0 D_{\mathbf{S}} a_0 = a_{\mathbf{S}} D_{\mathbf{S}} D_0 a_0 + a_{\mathbf{S}} (g F_{0\mathbf{S}} \times a_0)$. In the total derivatives we reduced the range of the indices \mathbf{R}, \mathbf{S} to three-dimensional ones: $r, s = 1, 2, 3$. Similarly,

$$\begin{aligned} \left[\frac{1}{4} F_{\mathbf{RS}}^2 \right]^{(2)} &= -\frac{1}{2} a_{\mathbf{S}} D_{\mathbf{R}}^2 a_{\mathbf{S}} + \frac{1}{2} a_{\mathbf{S}} D_{\mathbf{S}} D_{\mathbf{R}} a_{\mathbf{R}} + g F_{\mathbf{RS}} (a_{\mathbf{R}} \times a_{\mathbf{S}}) \\ &\quad + \frac{1}{2} \partial_r (a_{\mathbf{S}} D_r a_{\mathbf{S}}) - \frac{1}{2} \partial_r (a_{\mathbf{S}} D_{\mathbf{S}} a_r). \end{aligned} \quad (4.8)$$

Expanding accordingly the contributions from the gauge fixing term yields

$$\begin{aligned} T_{00}^{g.f.(2)} &= -2a_0 D_0 (D_{\mathbf{S}} a_{\mathbf{S}} - D_0 a_0) - \frac{1}{2} a_0 D_0^2 a_0 \\ &\quad + a_{\mathbf{S}} D_{\mathbf{S}} D_0 a_0 - \frac{1}{2} a_{\mathbf{R}} D_{\mathbf{R}} D_{\mathbf{S}} a_{\mathbf{S}} \\ &\quad - \frac{1}{2} \partial_r (a_r D_{\mathbf{S}} a_{\mathbf{S}}) - \frac{1}{2} \partial_0 (a_0 D_0 a_0) + \partial_0 (a_0 D_{\mathbf{S}} a_{\mathbf{S}}). \end{aligned} \quad (4.9)$$

Using that

$$a_{\mathbf{S}} D_{\mathbf{S}} D_0 a_0 - a_0 D_0 D_{\mathbf{S}} a_{\mathbf{S}} = -\partial_0 (a_0 D_{\mathbf{S}} a_{\mathbf{S}}) + \partial_s (a_s D_0 a_0) \quad (4.10)$$

we find for the final bulk terms, adding also the contributions from the fermions and the ghosts,

$$\begin{aligned} T_{00}^{(2)\text{bulk}} &= T_{00}^{(2)\text{ferm}} - \frac{1}{2} a_{\mathbf{S}} (D_0^2 + D_{\mathbf{R}}^2) a_{\mathbf{S}} + a_0 \left(\frac{3}{2} D_0^2 - \frac{1}{2} D_{\mathbf{S}}^2 \right) a_0 \\ &\quad + 2g F_{0\mathbf{S}} (a_0 \times a_{\mathbf{S}}) + g F_{\mathbf{RS}} (a_{\mathbf{R}} \times a_{\mathbf{S}}) \\ &\quad - \frac{1}{2} b D_0^2 c - \frac{1}{2} (D_0^2 b) c - \frac{1}{2} b D_{\mathbf{S}}^2 c - \frac{1}{2} (D_{\mathbf{S}}^2 b) c \\ &= T_{00}^{(2)\text{ferm}} + a_0 D_0^2 a_0 - a_{\mathbf{S}} D_0^2 a_{\mathbf{S}} + 2g F_{0\mathbf{S}} (a_0 \times a_{\mathbf{S}}) - b D_0^2 c - (D_0^2 b) c. \end{aligned} \quad (4.11)$$

In the last line we have used the linearized field equations, which in the background-covariant $\xi = 1$ gauge read

$$\begin{aligned} D_0^2 a_{\mathbf{R}} &= D_{\mathbf{S}}^2 a_{\mathbf{R}} + 2g F_{\mathbf{RS}} \times a_{\mathbf{S}} - 2g F_{\mathbf{R}0} \times a_0, \\ D_0^2 a_0 &= D_{\mathbf{S}}^2 a_0 + 2g F_{0\mathbf{S}} \times a_{\mathbf{S}}, \\ D_0^2 (b, c) &= D_{\mathbf{S}}^2 (b, c). \end{aligned} \quad (4.12)$$

For the total derivative terms we get

$$\begin{aligned} T_{00}^{(2)\text{tot.deriv.}} &= \frac{1}{2} \partial_0 (a_{\mathbf{S}} D_0 a_{\mathbf{S}}) + \frac{1}{2} \partial_s (a_0 D_s a_0) - \partial_0 (a_{\mathbf{S}} D_{\mathbf{S}} a_0) \\ &\quad + \frac{1}{2} \partial_r (a_{\mathbf{S}} D_r a_{\mathbf{S}}) - \frac{1}{2} \partial_r (a_{\mathbf{S}} D_{\mathbf{S}} a_r) - \frac{1}{2} \partial_r (a_r D_{\mathbf{S}} a_{\mathbf{S}}) \\ &\quad - \frac{1}{2} \partial_0 (a_0 D_0 a_0) + \partial_0 (a_0 D_{\mathbf{S}} a_{\mathbf{S}}) - 2\partial_0 (a_0 D_{\mathbf{S}} a_{\mathbf{S}}) + 2\partial_r (a_r D_0 a_0) \\ &\quad + \frac{1}{2} \partial_0^2 (bc) + \frac{1}{2} \partial_r^2 (bc) \end{aligned} \quad (4.13)$$

Since the expectation value of products of quantum fields is time-independent in a soliton background, we can drop total ∂_0 derivatives. Using furthermore $a_0 D_{\mathbf{S}} a_0 = \frac{1}{2} \partial_s a_0^2$ and $a_{\mathbf{S}} D_r a_{\mathbf{S}} = \frac{1}{2} \partial_r a_{\mathbf{S}}^2$, the result for the surface terms simplifies to

$$T_{00}^{(2)\text{tot.deriv.}} = \frac{1}{4} \partial_s^2 a_0^2 + \frac{1}{4} \partial_r^2 a_{\mathbf{S}}^2 + \frac{1}{2} \partial_r^2 (bc) - \frac{1}{2} \partial_r \partial_s (a_r a_s) + 2 \partial_r (a_r D_0 a_0). \quad (4.14)$$

In the special case of a monopole (rather than a dyon) background, the last term in (4.14) plays no role at the one-loop level, because in a monopole background there is no propagator between a_r and a_0 .

Let us now also specialize the bulk contribution to a monopole background. We then have $F_{0\mathbf{S}} = 0$ and $D_0 = \partial_0$. Furthermore, $F_{\mathbf{R}\mathbf{S}}$ is nonzero only when \mathbf{R}, \mathbf{S} are both from the set of spatial indices 1,2,3 plus the index corresponding to the Higgs field (6 for the $N = 2$ case, one of 5, 6, 7 for the $N = 4$ case). As in section 3.1 we denote indices from the latter set by $\underline{m}, \underline{n}$, so that $a_{\underline{m}} = (a_m, s)$. The bosonic linearized field equations (4.12) then decompose into

$$\begin{aligned} D_0^2 a_{\underline{m}} &= \partial_0^2 a_{\underline{m}} = D_{\underline{n}}^2 a_{\underline{m}} + 2g F_{\underline{mn}} \times a_{\underline{n}}, \\ \partial_0^2 (a_0, p, b, c; q_I) &= D_{\underline{s}}^2 (a_0, p, b, c; q_I). \end{aligned} \quad (4.15)$$

Here p is the pseudoscalar field, forming a quartet with a_0, b, c . For the fluctuations of the extra four scalar and pseudoscalar fields of the $N = 4$ theory we have introduced the notation q_I with $I = 1, \dots, 4$ for $N = 4$. This index I has the same range but of course otherwise nothing in common with the $SU(4)$ index on the dimensionally reduced Majorana spinors λ^I .

Explicitly, we have for the bulk contribution to the Hamiltonian density in the $N = 2$ case

$$T_{00}^{(2)\text{bulk}, N=2} = -a_{\underline{m}} \partial_0^2 a_{\underline{m}} + a_0 \partial_0^2 a_0 - p \partial_0^2 p - b \partial_0^2 c - (\partial_0^2 b) c + \frac{i}{2} \psi^\dagger \overleftrightarrow{\partial}_0 \psi, \quad (4.16)$$

and in the $N = 4$ case

$$\begin{aligned} T_{00}^{(2)\text{bulk}, N=4} &= -a_{\underline{m}} \partial_0^2 a_{\underline{m}} + a_0 \partial_0^2 a_0 - p \partial_0^2 p - b \partial_0^2 c - (\partial_0^2 b) c \\ &\quad - q_I \partial_0^2 q_I + \frac{i}{2} (\lambda^I)^T \partial_0 \lambda^I. \end{aligned} \quad (4.17)$$

The evaluation of the expectation value of T_{00} requires regularization. Our method is to use dimensional regularization, where we extend the range of the spatial indices to $1, \dots, 3 + \epsilon$, with $0 \leq \epsilon \leq 1$ and the extra dimension being chosen in one of the directions where the monopole background can be trivially embedded, i.e. one of the directions not involving the gauge field component A_I which has been chosen to accommodate the scalar field of the monopole (or the nonvanishing expectation value in the trivial sector). For definiteness we shall indicate the extra dimension from where we descend by continuous dimensional reduction by the index 5.

Inserting the mode expansions (3.16) and (3.18) in the expectation value of $T_{00}^{(2)\text{bulk}}$ and integrating over space¹⁷ one finds that it has the form of a sum over zero-point energies

$$\begin{aligned}
 M^{(1)\text{bulk}} &= \int \langle T_{00}^{(2)\text{bulk}} \rangle d^3x = \int d^3x \int \frac{d^3k d^\epsilon \ell}{(2\pi)^{3+\epsilon}} \frac{\omega}{2} (\mathcal{N}_+ |\chi_k^+|^2(x) + \mathcal{N}_- |\chi_k^-|^2(x)) \\
 &= \mathcal{N} \int d^3x \int \frac{d^3k d^\epsilon \ell}{(2\pi)^{3+\epsilon}} \frac{\omega}{2} (|\chi_k^+|^2(x) - |\chi_k^-|^2(x))
 \end{aligned}
 \tag{4.18}$$

where we recall that $\omega = \sqrt{\omega_k^2 + \ell^2}$. The factor $\mathcal{N} = \mathcal{N}^+ = -\mathcal{N}^-$ arises for the two cases of $N = 2$ and 4 as follows. For $N = 2$ we have $\mathcal{N}^+ = 4 - 2 = 2$ coming from $a_{\underline{m}}$ and ψ_+ , and $\mathcal{N}^- = 1 + 1 - 1 - 1 - 2 = -2$, coming from a_0, p, b, c , and ψ_- . For the $N = 4$ theory there is a complete cancellation, because then $\mathcal{N}^+ = 4 - 4 = 0$ and $\mathcal{N}^- = 1 + 1 - 1 - 1 + 4 - 4 = 0$, with the extra $+4$ in \mathcal{N}^- coming from the extra scalars q_I . The term with a_0 in (4.16) contributes the same way as p to the mass because the different sign in (4.16) is taken care of by the negative sign in $[a_0(k, \ell), a_0^\dagger(k', \ell')] = -\delta(k, k')\delta(\ell, \ell')$. For the ghosts one gets an extra minus sign because the propagator is $\langle cb \rangle$, not $\langle bc \rangle$, and b and c anticommute.

We note that in (4.18) any discrete modes can indeed be dropped — zero modes lead to scaleless integrals which vanish in dimensional regularization, whereas any massive modes for bound states cancel between fermions and bosons. Note also that (4.18) is well-defined only because of the difference $(|\chi_k^+|^2(x) - |\chi_k^-|^2(x))$. Treating the two contributions separately one could not easily integrate first over x and afterwards over momenta, as we shall do eventually. However, in the combined expression we can interchange the order of the integrations and use index theorem techniques to evaluate the spectral density

$$\Delta\rho(k^2) = \int d^Dx (|\chi_k^+|^2(x) - |\chi_k^-|^2(x)),
 \tag{4.19}$$

with $D = 3$ for the monopole model, but in the following we shall also consider the analogous situation for lower-dimensional solitons.

5. Index theorems and susy spectral densities

As we have seen, for susy solitons (and susy instantons as well) the linearized field equations for the quantum fields display a universal and simple pattern: one-half of the fermions have the same field equations as some of the bosons, whereas the field equations of the other half of the fermions coincides with those of the rest of the bosons and these differ only by an interaction term without derivatives. (If gauge fields are present, one may choose a gauge fixing term such that the field equations of the quartet of ghosts, antighosts, timelike gauge fields a_0 , and pseudoscalar field p are all equal.) Moreover, every solution of the first field equation with nonzero eigenvalue corresponds to a solution of the second field equation with the same eigenvalue, and vice-versa. Taking into account that for fermions the number of degrees of freedom is half the number of field components, one might conclude that for susy

¹⁷Spatial integration with respect to the extra ϵ dimensions would give simply a factor L^ϵ where L is the extent of the (trivial) extra dimension.

solitons and instantons the sum over nonzero modes of all bosons and fermions cancels.¹⁸ However, this conclusion is false in general. On a compact space it would be correct (with suitable boundary conditions), but on an open space the density of states of the first and the second field operator may be, and is in some cases, different. The continuous spectrum can be labeled by a vector \vec{k} which corresponds to the distorted plane waves of the scattering states. If these solutions of the field equations depend on time through a factor $e^{-i\omega t}$, the eigenvalues of the continuous spectrum of each field operator are equal to $\omega^2 = \vec{k}^2 + m^2$, where m is the mass of the quantum fields far away from the solitons. Let us denote the density of states of the first and the second field operator by $\rho^+(k^2)$ and $\rho^-(k^2)$, respectively. Then the sum of zero-point energies of the continuous spectrum can be written as

$$M^{(1)\text{bulk}} = \text{Tr} \frac{1}{2} \hbar \omega = \mathcal{N} \int \frac{1}{2} \hbar \omega \left(\rho^+(\vec{k}^2) - \rho^-(\vec{k}^2) \right) \frac{d^D \vec{k}}{(2\pi)^D} \quad (5.1)$$

in $D+1$ dimensional Minkowski spacetime, where \mathcal{N} depends on the number of components of the spinors. The difference $\Delta\rho = \rho^+(\vec{k}^2) - \rho^-(\vec{k}^2)$ vanishes in some cases such as the $N = 2$ vortex, the instanton, and the $N = 4$ monopole, but not in other cases such as the minimally supersymmetric kink and the $N = 2$ monopole, where it gives rise to anomalous contributions to the mass and central charge of these solitons.

In the case of the susy kink and the susy vortex, it is possible to evaluate $\Delta\rho$ rather easily [62] by rewriting it into a surface term after expressing $\chi_{\vec{k}}^-(\vec{x})$ in terms of $\chi_{\vec{k}}^+(\vec{x})$ by using the Dirac operator and by using the known asymptotic values of $\chi_{\vec{k}}^+(\vec{x})$. However, for the higher-dimensional cases, the mode expansions become progressively more complicated. One would prefer to deal with the fields themselves, instead of the mode functions. Here another method brings relief, revealing moreover an intriguing relationship of a nonvanishing $\Delta\rho$ to anomalies.

The basic idea is to start with the spatial components of the axial vector current $j_\mu = \bar{\psi} \gamma_5 \gamma_\mu \psi$ where the two halves of the components of ψ have field equations corresponding to the two sets under consideration. Taking the vacuum expectation value reduces the problem from $D+1$ dimensions to one in D spatial dimensions. The space-divergence of its vacuum expectation value, $\partial/\partial x^i \text{Tr} \gamma_5 \gamma_i \langle \psi(x) \bar{\psi}(y) \rangle$ can be written as a bulk integral involving $\Delta\rho$, but at the same time one can evaluate the space integral of $\partial_i j_i$ explicitly, because in a perturbation expansion in the inverse radius r^{-1} only a few terms contribute. Because the bulk integrals involving $\Delta\rho$ correspond to Feynman graphs which are power-counting divergent, one must regularize these integrals. In a series of papers, E. Weinberg has tackled this problem, using Pauli-Villars regularization for vortices [43] and monopoles [42]. For instantons, L.S. Brown et al. [63] have given a simple algebraic argument that $\Delta\rho = 0$. However, as we shall argue below, the same argument applies to kinks where we know already from a direct calculation that $\Delta\rho \neq 0$. In ref. [42] Weinberg has given an argument why $\Delta\rho = 0$ for instantons: the leading (logarithmic) divergence cancels due to the presence of the selfdual antisymmetric 't Hooft symbols. In the following we shall present a uniform

¹⁸For the quartet (a_0, b, c, p) this cancellation does indeed occur because the ghosts and antighosts contribute terms with a negative sign, whereas for a_0 and p the sign is positive.

treatment of solitons and instantons, which covers kinks, vortices, monopoles as well as instantons. From the point of view of this paper, the relevance of this analysis is that it yields the spectral density $\Delta\rho$ that is required in the evaluation of the bulk contributions to the quantum mass and central charge of susy monopoles.

The field operators for the fermionic modes in the soliton background without extra dimensions can be written in all cases in the following *generic* form (see (3.10) for the case of monopoles)

$$\mathcal{D}\psi_+ + i\omega\psi_- = 0, \quad \bar{\mathcal{D}}\psi_- + i\omega\psi_+ = 0. \quad (5.2)$$

Consider for example the $N = 1$ susy kink in 1+1 dimensions, or any solitonic model for that matter with a potential $V(\varphi) = \frac{1}{2}U^2(\varphi)$. It contains a real scalar field φ and a Majorana spinor¹⁹ $\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, and for a suitable representation of the Dirac matrices, namely $\gamma^0 = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\gamma^1 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we find for the Dirac equation

$$\mathcal{D}\psi_+ \equiv (\partial_x + U')\psi_+ = -i\omega\psi_-, \quad \bar{\mathcal{D}}\psi_- \equiv (\partial_x - U')\psi_- = -i\omega\psi_+ = 0. \quad (5.3)$$

Iterating the Dirac equation and using the BPS equation $\partial_x\varphi_K + U(\varphi_K) = 0$ one finds

$$\begin{aligned} -\bar{\mathcal{D}}\mathcal{D}\psi_+ &= (-\partial_x^2 + U'U' + UU'')\psi_+ = \omega^2\psi_+, \\ -\mathcal{D}\bar{\mathcal{D}}\psi_- &= (-\partial_x^2 + U'U' - UU'')\psi_- = \omega^2\psi_-, \end{aligned} \quad (5.4)$$

where U' and U'' are to be evaluated for the kink background φ_K . Clearly, ψ_+ and $\eta = \varphi - \varphi_K$ have the same linearized field equation.

The field equations for ψ_+ and ψ_- in the kink background can be written in 2×2 matrix form by introducing an operator \mathcal{P}

$$\mathcal{P} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} 0 & \bar{\mathcal{D}} \\ \mathcal{D} & 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = \begin{pmatrix} 0 & \partial_x - U' \\ \partial_x + U' & 0 \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \quad (5.5)$$

where

$$\mathcal{P} = \sigma^j \mathcal{D}_j \quad \text{with} \quad \mathcal{D}_1 = \partial_x, \quad \mathcal{D}_2 = -iU'. \quad (5.6)$$

The operator \mathcal{P} is anti-hermitian, $\mathcal{P}^\dagger = -\mathcal{P}$. Note that we are considering the operators \mathcal{D} and $\bar{\mathcal{D}}$ as acting in a one-dimensional space with coordinate x , although the spinor space is two-dimensional.²⁰

Index theorems are primarily used to compute the number of zero modes (normalizable solutions to the linearized fluctuation equations which are time-independent). For this purpose one considers

$$\mathcal{J}(M^2) = \text{Tr} \left(\frac{M^2}{-\bar{\mathcal{D}}\mathcal{D} + M^2} - \frac{M^2}{-\mathcal{D}\bar{\mathcal{D}} + M^2} \right) \quad (5.7)$$

¹⁹Note that in (3.9) we defined $\psi = \begin{pmatrix} \psi_+ \\ i\psi_- \end{pmatrix}$ for the monopole.

²⁰For the susy kink, ψ_+ and ψ_- are just the components of the spinor field, and we shall have the same situation in the case of the monopole, but for the vortex ψ_+ and ψ_- are linear combinations of the matter and gauge fermions of this model [39].

and defines the index of \mathcal{D} as $\mathcal{I} = \lim_{M^2 \rightarrow 0} \mathcal{J}(M^2)$. In the limit $M^2 \rightarrow 0$, only zero modes can contribute²¹ so that \mathcal{I} gives the difference of the number of zero modes of $\bar{\mathcal{D}}\mathcal{D}$ and $\mathcal{D}\bar{\mathcal{D}}$. In all models we consider, only $\bar{\mathcal{D}}\mathcal{D}$ has zero modes. E.g., for the kink with $U = (\lambda/2)^{1/2}(\varphi^2 - m^2/2\lambda)$, the potential $U'U' + UU'' = (\frac{1}{2}U^2)'' = 3\lambda\varphi^2 - m^2/2$ in $\bar{\mathcal{D}}\mathcal{D}$ has a zero mode corresponding to the position of the kink, but in $\mathcal{D}\bar{\mathcal{D}}$ one finds the positive definite potential $U'U' - UU'' = \lambda\varphi^2 + m^2/2$ without zero modes. Both potentials tend to m^2 as $|x| \rightarrow \infty$, which identifies m^2 as the asymptotic mass in terms of which $\omega^2 = k^2 + m^2$.

Since, with the exception of zero modes, all eigenvalues of $\bar{\mathcal{D}}\mathcal{D}$ and $\mathcal{D}\bar{\mathcal{D}}$ are the same, the quantity $\mathcal{J}(M^2)$ is directly related to the difference of the spectral densities of the continuum modes

$$\mathcal{J}(M^2) - \mathcal{J}(0) \equiv \mathcal{J}_{\text{cont.}}(M^2) = \int \frac{d^D \vec{k}}{(2\pi)^D} \frac{M^2}{\omega^2 + M^2} \Delta\rho(\vec{k}^2). \quad (5.8)$$

One can rewrite $\mathcal{J}(M^2)$ as a trace over a spinor space which is twice as large

$$\mathcal{J}(M^2) = \text{Tr} \left(\frac{M^2}{-\mathcal{D}^2 + M^2} \Gamma_5 \right) \quad (5.9)$$

where $\Gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the chirality operator, and $\mathcal{D} = \Gamma^j \mathcal{D}_j$ defines the Dirac matrices Γ^j . To denote the generic case we use capital Γ , which should not be confused with the higher-dimensional gamma matrices in (2.3) and (2.25). [For the example of the kink, $\Gamma_5 = \sigma^3$ and $\Gamma^j = \sigma^j$, see (5.6). Note that we first decomposed the Dirac equation into chiral parts, and then re-assembled these parts into nonchiral expressions using another representation for the Dirac matrices in (5.6). Also this is a general feature of the procedure.] To actually evaluate the continuum contribution to $\mathcal{J}(M^2)$, one introduces plane waves as bras and kets with $\langle \vec{k} | \vec{k}' \rangle = \delta^D(\vec{k} - \vec{k}')$,

$$\mathcal{J}(M^2) = \text{tr} \int d^D x \int d^D k \int d^D k' \langle \vec{x} | \vec{k} \rangle \langle \vec{k} | \frac{M^2}{-\mathcal{D}^2 + M^2} \Gamma_5 | \vec{k}' \rangle \langle \vec{k}' | \vec{x} \rangle, \quad (5.10)$$

where the trace tr is over spinor and group indices only. Then one pulls the ket plane wave $e^{-i\vec{k}' \cdot \vec{x}} / (2\pi)^{D/2}$ to the left, after which the derivatives ∂_j in \mathcal{D} are replaced by $\partial_j - ik'_j$, and $\langle k | k' \rangle = \delta^D(k - k')$ sets $k' = k$. The denominator can then be expanded as follows

$$(\vec{k}^2 + M^2) + \left(2i\vec{k} \frac{\partial}{\partial \vec{x}} - \mathcal{D}_j^2 - \frac{1}{2} \Gamma^j \Gamma^k \mathcal{F}_{jk} \right) \quad (5.11)$$

where $[\mathcal{D}_j, \mathcal{D}_k] = \mathcal{F}_{jk}$. (For example, for the kink one obtains

$$(k^2 + M^2) + \left(2ik \frac{\partial}{\partial x} - \partial_x^2 + U'U' + i\sigma^1 \sigma^2 \partial_x U' \right) \quad (5.12)$$

where $\partial_x U' = -U''U$ due to the BPS equation $\partial_x \varphi_K + U(\varphi_K) = 0$.)

At this point one encounters a technical problem which has a deep theoretical meaning. In order to explicitly evaluate $\mathcal{J}(M^2 \rightarrow 0)$ one first needs to know $\mathcal{J}(M^2)$ itself. If (as

²¹Provided the continuum spectrum is not too singular in the infrared, but this requirement is met in the cases we shall consider [42].

is the case in some models, but not in those of particular interest to us) $\mathcal{J}(M^2)$ is M^2 -independent, one can compute \mathcal{I} by taking instead the limit $M^2 \rightarrow \infty$ in $\mathcal{J}(M^2)$. In that case one can expand the denominator in

$$\mathcal{J}(M^2) = \int d^D x \int \frac{d^D k}{(2\pi)^D} \text{tr} \frac{M^2 \Gamma_5}{(\vec{k}^2 + M^2) + (2i\vec{k} \frac{\partial}{\partial \vec{x}} - \mathcal{D}_j^2 + \frac{1}{2} \Gamma^j \Gamma^k \mathcal{F}_{jk})} \quad (5.13)$$

around $\vec{k}^2 + M^2$, and only a few terms contribute for $M^2 \rightarrow \infty$. This is the calculation first performed by Fujikawa [64] to compute the chiral anomaly.

However, in some cases $\mathcal{J}(M^2)$ depends on M^2 . In these cases $\Delta\rho$ is nonvanishing. To still be able to compute \mathcal{J} , one may try to relate $\mathcal{J}(M^2)$ to a surface integral, because then instead of doing perturbation theory for $M^2 \rightarrow \infty$ one can hope to do perturbation theory for $r \rightarrow \infty$. At this point the axial anomaly comes to the rescue. Recalling that for massive fermions the axial current $j_5^\mu = \bar{\psi} \Gamma_5 \Gamma^\mu \psi$ satisfies

$$\partial_\mu j_5^\mu = -2M \bar{\psi} \Gamma_5 \psi + \text{anomaly} \quad (5.14)$$

and writing the fermion propagator as $\langle \psi(x) \bar{\psi}(y) \rangle = \frac{-i}{\not{p} + M} \delta(x - y)$, one finds for $\frac{i}{2}$ times the first term on the right-hand side, using $\text{tr} \bar{\psi} \Gamma_5 \psi = -\text{tr} \Gamma_5 \psi \bar{\psi}$,

$$\begin{aligned} iM \int d^D x \lim_{y \rightarrow x} \text{tr} \Gamma_5 \langle \psi(x) \bar{\psi}(y) \rangle &= M \text{Tr} \Gamma_5 \frac{1}{\not{p} + M} \\ &= \text{Tr} \Gamma_5 \frac{M^2}{-\not{p}^2 + M^2} = \mathcal{J}(M^2). \end{aligned} \quad (5.15)$$

We have then indeed an expression for $\mathcal{J}(M^2)$ as a total divergence plus an anomaly term.

Once again one is confronted with a new subtlety. When there is an anomaly, one should be careful and specify the regularization procedure. We use Pauli-Villars regularization with regulator mass μ , which is the most convenient for the technical step of extracting $\Delta\rho$. When the latter is used in the evaluation of the quantum corrections to the mass and central charge of the soliton, we shall switch to dimensional regularization and define our renormalization scheme in dimensional regularization. In the calculation of $\Delta\rho$ we do not have to renormalize yet.

We are thus led to consider the following expression [43]

$$J_i(x, y; M, \mu) = \text{Tr} \left(\langle x | \Gamma_5 \Gamma_i \frac{1}{\not{p} + M} | y \rangle - \langle x | \Gamma_5 \Gamma_i \frac{1}{\not{p} + \mu} | y \rangle \right) \quad (5.16)$$

This is the current we consider from now on; the fermion fields ψ and $\bar{\psi}$ themselves, in terms of which the axial current was defined in (5.14) will no longer be used. Note that the index i only takes on spacelike values. Hence, the whole analysis is in Euclidean space.

In all cases considered below we construct an operator \not{D} from operators \not{D} and $\bar{\not{D}}$ as in (5.5). These \not{D} consist of terms with ordinary derivatives $\Gamma^i \partial_i$ and terms K without these derivatives. From the off-diagonal structure of \not{D} it follows that Γ^i and K anticommute with the chirality operator Γ_5 . The coefficients Γ^i of ∂_i satisfy a Clifford algebra

$$\begin{aligned} \not{D} &= \Gamma^i \mathcal{D}_i \equiv \Gamma^i \partial_i + K, \\ \{\Gamma^i, \Gamma^j\} &= 2\delta^{ij}, \quad \{\Gamma^i, \Gamma_5\} = 0, \quad \{K, \Gamma_5\} = 0, \quad \Gamma_5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}. \end{aligned} \quad (5.17)$$

We need an identity for the Green function of $\mathcal{P} + M$; this identity appears in ref. [43], and we only enumerate here the steps one needs to derive it. One starts from

$$\begin{aligned} (\mathcal{P}_y + M)[\mathcal{P}_y + M]^{-1}\delta^D(x - y) &= \delta^D(x - y) \\ [\mathcal{P}_y + M]^{-1}(\mathcal{P}_y + M)\delta^D(x - y) &= \delta^D(x - y). \end{aligned} \quad (5.18)$$

Then one replaces $(\mathcal{P}_y + M)\delta^D(x - y)$ in the second identity by $\delta^D(x - y)(-\overleftarrow{\partial}_{x^i}\Gamma^i + K(x) + M)$, replaces the $\delta^D(x - y)$ in both relations by $\langle x|y\rangle$ and pulls the bra $\langle x|$ in the second relation to the left past $[\mathcal{P}_y + M]^{-1}$. Next one multiplies both relations by Γ_5 from the left, uses (5.17) to move Γ_5 past \mathcal{P}_y in the first relation, and takes the spinor trace. This yields

$$\begin{aligned} \frac{1}{2} \left(\frac{\partial}{\partial x^i} + \frac{\partial}{\partial y^i} \right) J^i(x, y; M, \mu) &= \\ &= -M \operatorname{tr} \langle x | \Gamma_5 [\mathcal{P}_y + M]^{-1} | y \rangle + \mu \operatorname{tr} \langle x | \Gamma_5 [\mathcal{P}_y + \mu]^{-1} | y \rangle \\ &\quad - \frac{1}{2} \operatorname{tr} [(K(x) - K(y)) \Gamma_5 \langle x | [\mathcal{P}_y + M]^{-1} - [\mathcal{P}_y + \mu]^{-1} | y \rangle]. \end{aligned} \quad (5.19)$$

In the regulated identity we let y tend to x , in which case the last term vanishes. Through (5.15) we finally find the desired identity which expresses $\mathcal{J}(M^2)$ and thus $\Delta\rho$ in terms of a surface integral and the axial anomaly,

$$-\frac{1}{2} \int d^D x \frac{\partial}{\partial x^i} J^i(x, x; M, \mu) = \mathcal{J}(M^2) - \mathcal{J}(\mu^2). \quad (5.20)$$

We shall evaluate $\mathcal{J}(\mu^2)$ for large μ^2 and $\int d^D x \frac{\partial}{\partial x^i} J^i$ for large $|\vec{x}|$. In an odd number of dimensions there is no axial anomaly, and we shall indeed find that for the kink and the monopole, where there is an odd number of spatial dimensions, $\mathcal{J}(\mu^2)$ vanishes for $\mu^2 \rightarrow \infty$. Any M -dependence of \mathcal{J} is then due to the M -dependence of the surface term.

To evaluate the anomaly $\mathcal{J}(\mu^2 \rightarrow \infty)$, we insert plane waves as explained before, to obtain

$$\begin{aligned} \mathcal{J}(\mu^2) &= \int d^D x \int \frac{d^D k}{(2\pi)^D} \operatorname{tr} \Gamma_5 \frac{\mu^2}{(k^2 + \mu^2) - L} \\ L &= -2ik_j \partial_j + \partial_j^2 + (\Gamma^i K + K \Gamma^i)(\partial_i - ik_i) + \Gamma^i (\partial_i K) + K^2. \end{aligned} \quad (5.21)$$

Expanding the denominator in terms of L , only a few terms contribute for $\mu^2 \rightarrow \infty$, and of these only one term survives after taking the trace over spinor indices.

To evaluate the surface integral, we use that in all cases considered the potential K^2 tends to a constant m^2 for $|\vec{x}| \rightarrow \infty$. We write then

$$\begin{aligned} J^i(x, x; M, \mu) &= \int \frac{d^D k}{(2\pi)^D} \left\{ \operatorname{tr} \Gamma_5 \Gamma^i (-\mathcal{P} + ik) [k^2 + M^2 + m^2 - \ell]^{-1} \right. \\ &\quad \left. - \operatorname{tr} \Gamma_5 \Gamma^i (-\mathcal{P}) [k^2 + \mu^2 + m^2 - \ell]^{-1} \right\} \end{aligned} \quad (5.22)$$

with $\ell = L - m^2 \equiv -2ik_j \partial_j + \partial_j^2 + (\Gamma^i K + K \Gamma^i)(\partial_i - ik_i) + C$. The matrix-valued functions $\Gamma^i K + K \Gamma^i$ and C tend to zero for large \vec{x} at least as fast as $|\vec{x}|^{-1}$, and one can evaluate the surface integral by expanding in terms of ℓ .

5.1 The susy kink

For the susy kink the operator \mathcal{D} was given in (5.6). Using (5.12) we obtain for the anomaly

$$\mathcal{J}(\infty) = \lim_{\mu^2 \rightarrow \infty} \int dx \int \frac{dk}{2\pi} \text{tr} \sigma_3 \mu^2 [(k^2 + \mu^2) + (2ik \frac{\partial}{\partial x} - \partial_x^2 + U'U' - \sigma_3 \partial_x U')]^{-1} \quad (5.23)$$

which vanishes as expected.

For the surface integral we obtain

$$\begin{aligned} -\frac{1}{2} \int dx \frac{\partial}{\partial x} J(x, x; M, \infty) &= -\frac{1}{2} \int dx \frac{\partial}{\partial x} \int \frac{dk}{2\pi} \text{tr} \frac{\sigma_3 \sigma_1 (-\sigma_1 \partial_x + i\sigma_2 U')}{k^2 + M^2 + m^2} \\ &= U' \Big|_{x=-\infty}^{x=\infty} \int \frac{dk}{2\pi} \frac{1}{k^2 + M^2 + m^2} = \frac{m}{\sqrt{m^2 + M^2}} \end{aligned} \quad (5.24)$$

The index is given by $\mathcal{I} = \mathcal{J}(0) = 1$, and the corresponding zero mode is the zero mode for translations.

The spectral density $\Delta\rho$ is determined by (5.8),

$$\mathcal{J}(M^2) - \mathcal{J}(0) = \frac{m}{\sqrt{m^2 + M^2}} - 1 = \int \frac{dk}{2\pi} \frac{M^2}{k^2 + m^2 + M^2} \Delta\rho(k^2). \quad (5.25)$$

One can solve this integral equation by a Laplace transform, and the result is [11, 15, 41]

$$\Delta\rho(k^2) = \frac{-2m}{k^2 + m^2}. \quad (5.26)$$

Thus for the susy kink $\mathcal{J}(M^2)$ is M -dependent and hence $\Delta\rho$ nontrivial. This shows that the argument given in ref. [63] which uses cyclicity of the trace to prove the M -independence of $\mathcal{J}(M^2)$ for instantons is inapplicable, because the kink provides a counter example. Because in the kink there are no long-range massless fields, also the explanation given in the appendix of ref. [42] of why the argument of ref. [63] does not apply to the case of monopoles is incomplete.

5.2 The $N = 2$ susy vortex

The $N = 2$ vortex model in 2+1 dimensions [65–67, 38, 39] contains one abelian gauge field, one complex scalar ϕ , another real scalar and one two-component complex gaugino and matter fermion. One can introduce linear combinations U and V of the gaugino and matter fermion, which can be viewed as the chiral parts of a complex four-component spinor

$$\mathcal{D}U = -i\omega V, \quad \bar{\mathcal{D}}V = -i\omega U \quad (5.27)$$

with

$$\mathcal{D} = \begin{pmatrix} D_+^V & -i\sqrt{2}e\phi_V \\ i\sqrt{2}e\phi_V^* & \partial_- \end{pmatrix}, \quad \bar{\mathcal{D}} = \begin{pmatrix} D_-^V & i\sqrt{2}e\phi_V \\ -i\sqrt{2}e\phi_V^* & \partial_+ \end{pmatrix}, \quad (5.28)$$

where $D_{\pm} = \partial_{\pm} - ieA_{\pm}$, $\partial_{\pm} = \partial_1 \pm i\partial_2$, $A_{\pm} = A_1 \pm iA_2$. Quantities with sub- or superscript V refer to the background fields of the vortex solution [68–71], $A_j^V(\vec{x})$ and $\phi_V(\vec{x})$ with $j = 1, 2$. Since $\mathcal{D}^\dagger = -\bar{\mathcal{D}}$ we introduce a 4×4 matrix-valued antihermitian operator \mathcal{P}

$$\mathcal{P} = \begin{pmatrix} 0 & \bar{\mathcal{D}} \\ \mathcal{D} & 0 \end{pmatrix} \equiv \Gamma^i \partial_i + K. \quad (5.29)$$

The operator $\mathcal{P}\mathcal{P}$ contains the operator $\bar{\mathcal{D}}\mathcal{D}$ and $\mathcal{D}\bar{\mathcal{D}}$ along the diagonal, respectively, where

$$\bar{\mathcal{D}}\mathcal{D} = \begin{pmatrix} (D_k^V)^2 - e^2(3|\phi_V|^2 - v^2) & -i\sqrt{2}e(D_-\phi_V) \\ i\sqrt{2}e(D_-\phi_V)^* & \partial_k^2 - 2e^2|\phi_V|^2 \end{pmatrix}, \quad (5.30)$$

with $k = 1, 2$ and

$$\mathcal{D}\bar{\mathcal{D}} = \begin{pmatrix} (D_k^V)^2 - e^2(|\phi_V|^2 + v^2) & 0 \\ 0 & \partial_k^2 - 2e^2|\phi_V|^2 \end{pmatrix}. \quad (5.31)$$

Only the operator (5.30) has zero modes, and this is also the operator which governs the fluctuations of the (complex) doublet $(\eta, a_+/\sqrt{2})$, where $\eta = \phi - \phi_V$ and $a_+ = A_+ - A_+^V$. For large $|\vec{x}|$, the operator $\mathcal{P}\mathcal{P}$ approaches the diagonal matrix $\{(D_k^V)^2 - m^2, \partial_k^2 - m^2, (D_k^V)^2 - m^2, \partial_k^2 - m^2\}$, $m^2 = 2e^2v^2$, exponentially fast.

We are now ready to compute the anomaly and the surface term for the vortex. There is an anomaly in two dimensions, and it is given by

$$\mathcal{J}(\mu^2) = \int \frac{d^2k}{(2\pi)^2} \frac{\mu^2}{(k^2 + \mu^2)^2} \int d^2x \operatorname{tr}(\Gamma_5 \Gamma^i \partial_i K), \quad (5.32)$$

because $\operatorname{tr} \Gamma_5 K^2 = -\operatorname{tr} K \Gamma_5 K = 0$. Since Γ_5 is diagonal, we only need the diagonal entries of $\Gamma^i \partial_i K$, which are easily read off from (5.29) and (5.28), and are given by $(-ie\partial_- A_+^V, 0, -ie\partial_+ A_-^V, 0)$. This leads to

$$\int d^2x ie(\partial_+ A_-^V - \partial_- A_+^V) = \int d^2x 2eF_{12} = 4\pi n, \quad (5.33)$$

where n is the winding number of the vortex background. Using $(2\pi)^{-2} \int d^2k (k^2 + \mu^2)^{-2} = (4\pi\mu^2)^{-1}$, we find $\mathcal{J}(\mu^2) = n$.

The M -dependent part of the surface term is given by

$$-\frac{1}{2} \int d^2x \partial_i \mathcal{J}^i = \int d^2x \partial_i \int \frac{d^2k}{(2\pi)^2} \left[(k^2 + M^2 + m^2)^{-1} \operatorname{tr} \Gamma_5 \Gamma^i (-K) + (k^2 + M^2 + m^2)^{-2} \operatorname{tr} \Gamma_5 \Gamma^i (-\Gamma^j \partial_j - K) C \right]. \quad (5.34)$$

The last term does not contribute since all entries of C tend to zero exponentially fast as $r \rightarrow \infty$ [69–71]. The first term also vanishes: according to Stokes' theorem $\int d^2x \partial_i v^i = \int d^2x g^{-1/2} \partial_i g^{ij} g^{1/2} v_j = \int d\theta x^i v_i$, but the trace $\operatorname{tr} \Gamma_5 x^i \Gamma_i K$ is proportional to $x^+ A_-^V - x^- A_+^V$ (replace ∂_i by x^i in the step from (5.32) to (5.33)), and this vanishes (see (2.9)).

The final result for \mathcal{J} is therefore $\mathcal{J}(M^2) = n$ independently of M^2 . As we have discussed before, this implies that $\Delta\rho$ vanishes for the vortex. For the index we evidently have $\mathcal{I} = n$. Because the operator (5.30) is the fluctuation operator of $(\eta, a_+/\sqrt{2})$, this implies that there are n independent complex zero modes for ψ^+ , or $2n$ real zero modes for $(\operatorname{Re}\eta, \operatorname{Im}\eta, a_1, a_2)$ [35]. These correspond to the positions in the x - y plane of n simple vortices. (The zero modes associated with rotations of the vortex solutions are not new zero modes, but linear combinations of the translational ones; one can prove this in the same way as for instantons [72]).

5.3 The $N = 2$ susy monopole

For calculating $\Delta\rho$ for the $N = 2$ monopole, regularization is not needed, but we keep the Pauli-Villars contributions as a check. The fermionic fluctuation equations have again the form (5.2), with \mathcal{D} and $\bar{\mathcal{D}}$ given in (3.10), and we construct again the 4×4 matrix \mathcal{D} , whose decomposition $\mathcal{D} = \Gamma^i \partial_i + K$ defines K and Γ^i , satisfying (5.17). There are now two new features: (i) because the square of the scalar triplet S^2 tends to v^2 for large r as slowly as $S^2 = v^2(1 - 2(mr)^{-1} + \dots)$ (see (2.9)), there will be surface contributions from these scalars; (ii) the propagator $(-\mathcal{D}\mathcal{D} + M^2)^{-1}$ must be decomposed into two separate isospin sectors.

The anomaly term is given by (5.21) in the limit $\mu^2 \rightarrow \infty$, but this should vanish, because there are no chiral anomalies in odd (three) dimensions. (Anomalies will however appear when we later use the spectral densities to evaluate the quantum corrections to mass and central charge of the monopole.)

The surface term is given by (5.20) with (5.22), which reads more explicitly as follows

$$\begin{aligned}
 -\frac{1}{2} \int d^3x \partial_i J^i(x, x; M, \mu) &= -\frac{1}{2} \oint d\Omega r^2 \int \frac{d^3k}{(2\pi)^3} \text{tr} \left\{ \hat{x}^i \Gamma_5 \Gamma^i (i\Gamma^j k_j - \mathcal{D})^{ab} \right. \\
 &\quad \left. [(k^2 + M^2 + m^2 - \ell_+)^{-1} (\delta^{ca} - \hat{x}^c \hat{x}^a) + (k^2 + M^2 + m^2 - \ell_+)^{-1} \hat{x}^c \hat{x}^a] \right\}_{r \rightarrow \infty} \\
 &\quad - \{M^2 \rightarrow \mu^2\}
 \end{aligned} \tag{5.35}$$

where the trace is over spinor indices, and

$$\begin{aligned}
 \ell_-^{ab} &= -2ik_j \partial_j \delta^{ab} + (D_j^2)^{ab} + (g^2 S^2 - m^2) \delta^{ab} \\
 \ell_+^{ab} &= \ell_-^{ab} + \frac{1}{2} g \Gamma^{mn} F_{mn}^{ab}
 \end{aligned} \tag{5.36}$$

with $F_{mn}^{ab} = g \epsilon^{ac} F_{mn}^c$ and $F_{mn} = \begin{pmatrix} F_{mn} & D_m S \\ -D_n S & 0 \end{pmatrix}$. The Dirac matrices which appear in these expressions are given by

$$\Gamma^{\underline{m}} = \begin{pmatrix} 0 & \bar{\sigma}^{\underline{m}} \\ \sigma^{\underline{m}} & 0 \end{pmatrix}, \quad \Gamma^1 \Gamma^2 \Gamma^3 \Gamma^4 = -\Gamma^5, \quad \Gamma^5 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \tag{5.37}$$

and we have rewritten the term $\frac{1}{2} \bar{\sigma}^{\underline{mn}} F_{mn}$ from (3.12) as the 4×4 matrix $\frac{1}{2} \Gamma^{\underline{mn}} F_{mn}$ in order to deal with only one kind of Dirac matrices. The curvature F_{mn} falls off as $1/r^2$, and coming from the expansion of ℓ_+ is the only term which contributes. Inserting the asymptotic values of the monopole background field (2.9) we find for the M^2 -dependent part

$$\begin{aligned}
 \mathcal{J}(M^2) &= \lim_{r \rightarrow \infty} \int d\Omega r^2 \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2 + M^2 + m^2)^2} \\
 &\quad \times \text{tr} \frac{1}{2} \Gamma_5 \hat{x}^i \Gamma^i \Gamma^4 i S^{ab} \left(\frac{1}{2} g \Gamma^{jk} F_{jk} \right)^{ba} = \frac{2m}{\sqrt{M^2 + m^2}}.
 \end{aligned} \tag{5.38}$$

(The momentum integral yields $(8\pi \sqrt{M^2 + m^2})^{-1}$ and the trace tends to $4m/r^2$ as $r \rightarrow \infty$.) The contribution due to the Pauli-Villars regulator is proportional to $(\mu^2 + m^2)^{-1/2}$ and indeed vanishes for $\mu^2 \rightarrow \infty$.

The result for the index is $\mathcal{I} = \mathcal{J}(0) = 2$, corresponding to 2 complex zero modes for the fermions, and 4 real zero modes for a monopole with winding number unity: 3 zero modes for its position, and one for gauge orientations with respect to the unbroken $U(1)$. (The latter is given by $\delta A_i = (D_i S)\alpha$ and $\delta S = 0$ with α the modulus [42]).

Finally we extract the result for $\Delta\rho$ from (5.38). According to (5.8) we must invert

$$\frac{2m}{\sqrt{M^2 + m^2}} - 2 = \int \frac{d^3k}{(2\pi)^3} \frac{M^2}{k^2 + m^2 + M^2} \Delta\rho_{\text{mon}}(k^2). \quad (5.39)$$

The solution is

$$\Delta\rho_{\text{mon}}(k^2) = \frac{-4\pi m}{k^2(k^2 + m^2)}. \quad (5.40)$$

6. Results for the quantum monopole mass

In section 4 we have shown that the contributions to the mass of the $N = 2$ and $N = 4$ monopole consist of bulk terms which correspond to the familiar sum over zero-point energies of the bosonic and fermionic quantum fluctuations, and surface terms whose presence seems at first sight unusual and puzzling. As we shall see later in this section, the latter have the potential to contribute nontrivially to the one-loop quantum mass of monopoles, in contrast to lower-dimensional solitons. First, however, we shall use the results of the previous section to evaluate the quantum corrections to the monopole mass from the sum over zero-point energies [35].

6.1 Bulk contributions

In (4.18) we have found that in dimensional regularization the contributions from $T_{00}^{(2)\text{bulk}}$ have the form

$$M^{(1)\text{bulk}} = \int \langle T_{00}^{(2)\text{bulk}} \rangle d^3x = \mathcal{N} \int d^3x \int \frac{d^3k d^\ell \ell}{(2\pi)^{3+\epsilon}} \frac{\sqrt{k^2 + \ell^2 + m^2}}{2} \Delta\rho(k) \quad (6.1)$$

where $\mathcal{N} = 4 - N$ for the $N = 2$ and $N = 4$ theory, and $\Delta\rho$ given by (5.40).

Clearly $M^{(1)\text{bulk}}$ is logarithmically divergent. Carrying out the ℓ -integration we obtain²²

$$M^{(1)\text{bulk}} = -(4 - N) \frac{m}{\pi} \frac{\Gamma(-\frac{1}{2} - \frac{\epsilon}{2})}{(2\pi^{\frac{1}{2}})^\epsilon \Gamma(-\frac{1}{2})} \int_0^\infty dk (k^2 + m^2)^{-\frac{1}{2} + \frac{\epsilon}{2}} = -(4 - N) \frac{8\pi m}{1 + \epsilon} I \quad (6.2)$$

where I is the divergent integral introduced in (3.21).

The classical mass of a monopole is given by $M^{\text{cl}} = 4\pi m_0/g_0^2$. In the renormalization scheme given by (3.21), where $m = m_0$ and $g_0^{-2} - g^{-2} = 2(4 - N)I$, we have the counterterm

$$\delta M \equiv 4\pi m(g_0^{-2} - g^{-2}) = 8(4 - N)\pi m I. \quad (6.3)$$

Adding (6.2) and (6.3) we find that the divergent contributions from $I^{\text{div}} = -\frac{1}{8\pi^2} \frac{1}{\epsilon}$ cancel, but there is a finite remainder for $N = 2$,

$$M^{(1)\text{bulk}} + \delta M = -(4 - N) \frac{m}{\pi} + O(\epsilon), \quad (6.4)$$

²²Use $\int d^n p / (p^2 + M^2)^\alpha = \pi^{n/2} (M^2)^{n/2 - \alpha} \Gamma(\alpha - n/2) / \Gamma(\alpha)$.

which has been overlooked in refs. [36, 37], claiming vanishing quantum corrections. While the renormalization conditions of refs. [36, 37] were identical to the ones we specified in section 3.2, ref. [36] did not specify the regularization method used to obtain its null result, and ref. [37] regularized by inserting slightly different oscillatory factors in the two-point function and in the integral over the spectral density in (6.2), a procedure that is not evidently self-consistent.

As we shall discuss in section 7, the finite remainder (6.4) is associated with an anomalous contribution to the $N = 2$ central charge of equal magnitude. The renormalization scheme set up in section 3.2 is special in that in this scheme all of the quantum correction to the monopole mass are equal to the scheme-independent anomalous contribution of the central charge, whereas for other schemes one could also have further nonanomalous contributions to both mass and central charge. However, before we can conclude that (6.4) is the final result for the quantum corrections to the susy monopole mass, we have to discuss also the surface contributions.

6.2 Surface contributions

The surface terms for the susy monopole were obtained in (4.14); they contain no contributions from fermions, only from the vector, scalar, and ghost fields

$$M^{(1)\text{surf}} = \int d^3x T_{00}^{(2)\text{tot.deriv.}} = \int d^3x \left[\frac{1}{4} \partial_j^2 \langle a_0^2 + a_{\mathbf{S}}^2 + 2bc \rangle - \frac{1}{2} \partial_j \partial_k \langle a_j a_k \rangle + 2 \partial_j \langle a_j \partial_0 a_0 \rangle \right], \quad (6.5)$$

where $j, k = 1, 2, 3$. The difference between $N = 2$ and $N = 4$ is only the range of the index \mathbf{S} : $\mathbf{S} = 1, 2, 3, 5, 6$ for $N = 2$, and $\mathbf{S} = 1, 2, 3, 5, \dots, 10$ for $N = 4$. We shall show that the sum of all surface contributions cancels for the $N = 2$ monopole, but for $N = 4$ the extra four (pseudo-)scalars yield a new type of divergence which we will have to dispose of.

To evaluate (6.5) we need the propagators for the various fields in the monopole background. As one can see from the linearized bosonic fluctuation equations (4.15), the propagators for the fields $a_{\underline{m}} = (a_m, s)$ are given by

$$\langle a_{\underline{m}}^b(x) a_{\underline{n}}^c(y) \rangle = i \langle x | [(-\partial_0^2 \delta^{bc} + (D_{\underline{k}}^2)^{bc}) \delta_{\underline{mn}} + 2g\epsilon^{bac} F_{\underline{mn}}^a]^{-1} | y \rangle \quad (6.6)$$

whereas the remaining (pseudo-)scalar fields (p for $N = 2$; p and q_I for $N = 4$) all have the same propagator

$$\langle p^b(x) p^c(y) \rangle = i \langle x | [-\partial_0^2 \delta^{bc} + (D_{\underline{k}}^2)^{bc}]^{-1} | y \rangle, \quad (6.7)$$

As explained below (4.18), for a_0 we have

$$\langle a_0^b(x) a_0^c(y) \rangle = -\langle p^b(x) p^c(y) \rangle, \quad (6.8)$$

and the anticommutative nature of the ghosts leads to

$$\langle b^b(x) c^c(y) \rangle = -\langle c^c(y) b^b(x) \rangle = -\langle p^b(x) p^c(y) \rangle. \quad (6.9)$$

The covariant derivatives $(D_{\underline{k}}^2)^{ac} = (D_k^2)^{ac} + (gS \times (gS \times))^{ac}$ have a complicated form when written out

$$(D_{\underline{k}}^2)^{ac} = \partial_k^2 \delta^{ac} + 2\epsilon^{abc} g A_k^b \partial_k + g^2 (A_j^a A_j^c - \delta^{ac} A_j^b A_j^b) + g^2 (S^a S^c - \delta^{ac} S^2). \quad (6.10)$$

However, we only need the asymptotic values of the propagators. Substituting the asymptotic values (2.9) of the background fields we obtain

$$\begin{aligned} (D_{\underline{k}}^2)^{ab} &\rightarrow \partial_k^2 \delta^{ab} + \frac{2}{r} (\hat{x}^a \partial^b - \hat{x}^b \partial^a) - \frac{1}{r^2} (\delta^{ab} + \hat{x}^a \hat{x}^b) \\ &\quad - m^2 \left(1 - \frac{1}{mr}\right)^2 (\delta^{ab} - \hat{x}^a \hat{x}^b) \end{aligned} \quad (6.11)$$

where we recall that $m = gv$ and $\hat{x}^i \equiv x^i/r$. The term $gF_{\underline{mn}}^a$ appearing additionally in (6.6) has the asymptotic behavior given in (2.9)

$$gF_{\underline{mn}}^a \rightarrow -\epsilon_{mnk} \frac{\hat{x}^a \hat{x}^k}{r^2}, \quad gF_{m6}^a = -gF_{6m}^a = g(D_m S)^a \rightarrow -\frac{\hat{x}^a x^m}{r^2}. \quad (6.12)$$

Because the propagator $\langle a_j a_0 \rangle$ vanishes and thus all nontrivial surface terms in (6.5) involve two spatial derivatives, transforming (6.5) into an integral over the sphere at infinity we have

$$\int d^3x \partial_i \partial_j \dots = \lim_{r \rightarrow \infty} r^2 \oint d\Omega \hat{x}_i \partial_j \dots, \quad \int d^3x \partial_j^2 \dots = \lim_{r \rightarrow \infty} r^2 \oint d\Omega \frac{\partial}{\partial r} \dots \quad (6.13)$$

Hence, we need only keep the terms of order $1/r$ when expanding the propagators around their common leading term

$$(-\partial_0^2 + D_{\underline{k}}^2)^{-1ab} \rightarrow \frac{1}{\square - m^2} (\delta^{ab} - \hat{x}^a \hat{x}^b) + \frac{1}{\square} \hat{x}^a \hat{x}^b + O\left(\frac{1}{r}\right). \quad (6.14)$$

This straight away disposes of the terms with $F_{\underline{mn}}$. Moreover, we can also drop the terms proportional to $\frac{1}{r} (\hat{x}^a \partial^b - \hat{x}^b \partial^a)$ because the derivatives ∂^a and ∂^b either act on a factor $\frac{1}{r}$ and then produce terms falling off like $1/r^2$ or they produce a term proportional to a single loop momentum in the numerator (see below) in which case symmetric integration in momentum space gives a vanishing result. Hence, all propagators can be simplified to essentially the same form

$$\langle a_M^a(x) a_N^b(y) \rangle \rightarrow \eta_{MN} G^{ab}(x, y), \quad \langle b^a(x) c^b(y) \rangle \rightarrow -G^{ab}(x, y) \quad (6.15)$$

with

$$G^{ab}(x, y) = \langle x | \left[\frac{-i}{-\square + m^2 - \frac{2m}{r}} (\delta^{ab} - \hat{x}^a \hat{x}^b) + \frac{i}{\square} \hat{x}^a \hat{x}^b \right] | y \rangle. \quad (6.16)$$

With these results at hand, we find for the contribution of the surface terms to the mass of the $N = 2$ monopole

$$\begin{aligned} M^{(1)\text{surf}, N=2} &= \lim_{r \rightarrow \infty} \frac{1}{4} 4\pi r^2 \frac{\partial}{\partial r} \langle a_0^2 + a_j^2 + s^2 + p^2 + 2bc - 2s^2 \rangle \\ &= (-1 + 3 + 1 + 1 - 2 - 2) \lim_{r \rightarrow \infty} \pi r^2 \frac{\partial}{\partial r} \langle s^2 \rangle = 0. \end{aligned} \quad (6.17)$$

Thus in the $N = 2$ case the contributions from the surface terms in the quantum Hamiltonian cancel completely.

On the other hand, in the $N = 4$ case $\langle a_{\mathbb{S}}^2 \rangle$ in (6.5) involves four extra scalar fields and there is no longer a cancellation of these surfaces terms. We instead find

$$M^{(1)\text{surf},N=4} = 4 \times \frac{1}{4} \lim_{r \rightarrow \infty} 4\pi r^2 \frac{\partial}{\partial r} \langle s^2 \rangle \quad (6.18)$$

so that we have to evaluate the expression on the right-hand side. Up to an irrelevant constant we find in the limit of large r

$$\langle s^2 \rangle = G^{aa}(y, x)|_{y=x} \simeq 2 \langle y | \frac{-i}{-\square + m^2 - \frac{2m}{r}} | x \rangle |_{y=x} + \text{const.}, \quad (6.19)$$

where the factor 2 is due to the trace of $(\delta^{ab} - \hat{x}^a \hat{x}^b)$. Following the procedure of section 5 to evaluate such matrix elements by inserting complete sets of plane wave states we obtain

$$\begin{aligned} \langle s^2 \rangle &\simeq 2 \int \frac{d^{4+\epsilon} k}{(2\pi)^{4+\epsilon}} \frac{-i}{(k^2 + m^2) + 2ik^\mu \partial_\mu - \partial_\mu^2 - \frac{2m}{r}} \\ &\simeq 2 \left(\frac{2m}{r} \right) \int \frac{d^{4+\epsilon} k}{(2\pi)^{4+\epsilon}} \frac{-i}{(k^2 + m^2)^2} = 4 \frac{m}{r} I \end{aligned} \quad (6.20)$$

with I the logarithmically divergent integral defined previously in (3.21). Hence,

$$M^{(1)\text{surf},N=4} = -16\pi m I. \quad (6.21)$$

In the $N = 4$ theory there is no wave function or coupling constant renormalization that could produce a counterterm from the renormalization of the classical monopole mass. However, as we have mentioned in section 3.2, there is in general the need for (additive) composite-operator renormalization. The $N = 2$ theory is special in that the susy current multiplet is finite in the sense that standard wave function and coupling constant renormalization suffices. In the $N = 4$ theory, however, the unimproved susy current multiplet receives the counterterm (3.26) involving improvement terms. Since the expression for T_{00} that we have derived in section 4 is unimproved, we have additive renormalization through

$$\delta T_{00}^{\text{comp.op.ren.}} = -(Z_{\Delta T} - 1) \Delta T_{00}^{\text{impr}} \quad (6.22)$$

with $Z_{\Delta T} = Z_{\Delta U}$ and the latter given by (3.24). According to (2.29), the improvement term for the energy density reads

$$\Delta T_{00}^{\text{impr}} = -\frac{1}{6} \partial_j^2 (A_{\mathcal{J}} A^{\mathcal{J}}) \quad (6.23)$$

with $\mathcal{J} = 5 \dots 10$. This gives

$$\begin{aligned} \delta M^{\text{comp.op.ren.}} &= -(Z_{\Delta U} - 1) \int d^3 x \left(-\frac{1}{6} \right) \partial_j^2 (A_{\mathcal{J}} A^{\mathcal{J}}) \\ &= -(Z_{\Delta U} - 1) \int d^3 x \left(-\frac{1}{6} \right) \partial_j^2 (S^2) \\ &= \lim_{r \rightarrow \infty} -(Z_{\Delta U} - 1) \left(-\frac{1}{6} \right) 4\pi r^2 \frac{\partial}{\partial r} v^2 \left(1 - \frac{2}{mr} \right) \\ &= -(12g^2 I) \left(-\frac{4\pi m}{3g^2} \right) = +16\pi m I, \end{aligned} \quad (6.24)$$

which completely cancels (6.21).

As we have remarked in section 3.2, the improved susy current multiplets do not require composite-operator renormalization. Indeed, had we included the surface terms (6.23) in (6.5), we would have found complete cancellation in (6.18), because the improvement term involves 6 scalar fields in $N = 4$ and an overall factor $-\frac{1}{6}$.

Note that in $N = 2$ we had complete cancellation of the surface term contributions already in the unimproved energy density. In this model, and only there, the improvement terms themselves are finite operators. With \mathcal{J} running only over two values, we have

$$\int d^3x \left\langle -\frac{1}{6} \partial_j^2 a_{\mathcal{J}}^2 \right\rangle = -2 \times \frac{1}{6} \lim_{r \rightarrow \infty} 4\pi r^2 \frac{\partial}{\partial r} \langle s^2 \rangle = +\frac{16}{3} \pi m I \quad (6.25)$$

but this is now compensated by coupling constant and wave function renormalization, since with the value of Z_S , eq. (3.22), in the Feynman- R_ξ gauge we also have

$$(Z_S - 1) \int d^3x \left(-\frac{1}{6} \right) \partial_j^2 (S^2) = (4g^2 I) \left(-\frac{4\pi m}{3g^2} \right) = -\frac{16}{3} \pi m I. \quad (6.26)$$

To summarize, we have found that the quantum corrections from the surface terms in the Hamiltonian to the mass of the monopole cancel completely upon composite-operator renormalization, which is nontrivial in $N = 4$. The net result for the one-loop quantum mass is then given by the bulk contributions corresponding to the sum over zero-point energies plus a standard counterterm from the classical monopole mass, eq. (6.4), which is nontrivial only in $N = 2$,

$$M_{N=2}^{(1)} = -\frac{2m}{\pi}; \quad M_{N=4}^{(1)} = 0. \quad (6.27)$$

7. Central charge

In the previous sections we obtained for the $N = 2$ model a (negative) finite one-loop correction to the monopole mass from the bulk (6.4), whereas for the $N = 4$ model we found divergent one-loop surface contributions. The latter were cancelled by a composite operator renormalization which is proportional to the improvement term for the energy momentum tensor.

The non-vanishing corrections raise the question whether and how they agree with BPS saturation. Naively one would expect that the central charge (2.15), (2.28) as a topological object receives no quantum correction. The situation is however more complicated: both in $N = 2$ and in $N = 4$ a direct one-loop calculation for the central charge operator (2.15), (2.28) gives [37, 35]

$$\begin{aligned} U^{(1)} &= U^{(cl)} + \frac{1}{2} \int d^3x \partial_i (S^a \epsilon_{ijk} \langle F_{jk}^a [A + a] - F_{jk}^a [A] \rangle) \\ &= U^{(cl)} + \frac{1}{2} \int d^3x \epsilon_{ijk} \epsilon_{abc} \partial_i (g S^a \langle a_j^b a_k^c \rangle) \\ &= \frac{4\pi m}{g_0^2} - 16\pi m I, \end{aligned} \quad (7.1)$$

where I was given in (3.21). This result is due to a $\langle a_i a_j \rangle$ loop, which is evaluated using (6.6) and steps similar to the ones leading to (6.21), but now it is the same for the $N = 2$ and $N = 4$ model. According to (3.21) the coupling constant renormalizes as $\frac{1}{g_0^2} - \frac{1}{g^2} = 2(4 - N)I$, so that in the $N = 2$ case there is no one-loop correction, whereas in the $N = 4$ model the UV-divergence in (7.1) remains uncanceled.

$N = 4$ monopole. In the $N = 4$ model a detailed analysis showed that the operator U needs a composite operator renormalization which is proportional to the improvement term (2.34) [40]:

$$\delta U^{\text{comp.op.ren.}} = 4g^2 I \left[U + \frac{i}{8} \int d^3x \partial_i (\varepsilon^{ijk} \bar{\lambda} \alpha^1 \gamma_{jk} \lambda) \right]. \quad (7.2)$$

Taking this counterterm into account in (7.1) one obtains a vanishing one-loop correction for the central charge of the $N = 4$ monopole. This observation resolved the puzzle of an apparent divergent one-loop correction to the central charge [40]. Together with the results of section 6.2 this restores not only the BPS saturation $M = U$ at the one-loop order but also the super multiplet structure of the generators of the susy algebra. Both energy momentum tensor and central charge receive composite operator renormalizations proportional to improvement terms (6.22), (7.2) through one-loop surface terms. These composite operator counterterms form a susy multiplet [40].

$N = 2$ monopole. For the $N = 2$ monopole the situation is more complicated. There is no way to balance the finite mass correction $\Delta M^{(1)} = -\frac{2m}{\pi}$ we obtained in (6.4) by a one-loop contribution from (7.1). In [35] we have shown that the central charge despite its topological nature receives a one-loop correction which is associated with an anomaly of the (conformal) central charge current.

The anomaly in the (conformal) central charge turns out to be a member of the anomaly multiplet besides the trace and the super conformal anomaly. In our approach, where the theory is dimensionally regularized by embedding it in a higher dimensional space, this anomalous contribution to the central charge appears as a non-vanishing momentum flow of fermions in the regulating extra dimension. Due to quantum effects the violation of parity in the regulating five dimensional space has a finite reminder as an anomalous central charge contribution in the four dimensional world.

According to the choice (3.8) the central charge U stems from the T^{05} component of the six-dimensional “energy momentum” tensor, including the symmetric and anti-symmetric part of the r.h.s. of (2.13) in this notion²³. The antisymmetric part gave the ordinary central charge U . From the symmetric part, i.e. the genuine momentum in the regulating extra dimension, one thus gets potentially non-vanishing contribution to the central charge (2.15).

The mode functions and thus the propagators of the bosonic fields (3.3) are even in the extra momentum and therefore give no contributions to T^{05} . This is not the case for the

²³Using $\Gamma_7 \Gamma^{PQMRST} = \varepsilon^{PQMRST} \Gamma_T$ all terms of the r.h.s. of (2.13) are proportional to a single gamma matrix.

fermionic fields and thus propagators (3.7). Using the mode expansion (3.16) we obtain

$$\begin{aligned} \langle T_{05}^{\text{ferm}} \rangle &= \langle \bar{\psi} \gamma_0 \partial_5 \psi \rangle \\ &= - \int \frac{d^\epsilon \ell}{(2\pi)^\epsilon} \int \frac{d^3 k}{(2\pi)^3} \frac{\ell^2}{2\omega} (|\chi_k^+|^2 - |\chi_k^-|^2)(x) \end{aligned} \quad (7.3)$$

Here we have omitted terms linear in the extra momentum ℓ , as they vanish by symmetric integration, and contributions from massless modes (zero modes), which give scale-less integrals that do not contribute in dimensional regularization. Integration over space again leads to the difference of spectral densities $\Delta\rho$ defined in (4.19) and evaluated in (5.40). The anomalous contribution to the central charge thus becomes

$$\begin{aligned} U_{\text{anom}} &= \int d^3 x \langle \Theta_{05} \rangle = \int \frac{d^3 k d^\epsilon \ell}{(2\pi)^{3+\epsilon}} \frac{\ell^2}{2\sqrt{k^2 + \ell^2 + m^2}} \Delta\rho(k^2) \\ &= -4m \int_0^\infty \frac{dk}{2\pi} \int \frac{d^\epsilon \ell}{(2\pi)^\epsilon} \frac{\ell^2}{(k^2 + m^2)\sqrt{k^2 + \ell^2 + m^2}} \\ &= -8 \frac{\Gamma(1 - \frac{\epsilon}{2}) m^{1+\epsilon}}{(4\pi)^{1+\frac{\epsilon}{2}} 1 + \epsilon} = -\frac{2m}{\pi} + O(\epsilon). \end{aligned} \quad (7.4)$$

The correction (7.4) matches exactly the mass correction (6.4) so that the BPS bound is saturated at the one-loop level, but in a nontrivial way.

The nonzero result (7.4) is in fact in complete accordance with the low-energy effective action for $N = 2$ super-Yang-Mills theory as obtained by Seiberg and Witten [5–7].²⁴ According to the latter, the low-energy effective action is fully determined by a prepotential $\mathcal{F}(A)$, which to one-loop order is given by

$$\mathcal{F}_{1\text{-loop}}(A) = \frac{i}{2\pi} A^2 \ln \frac{A^2}{\Lambda^2}, \quad (7.5)$$

where A is a chiral superfield and Λ the scale parameter of the theory generated by dimensional transmutation. The value of its scalar component a corresponds in our notation to $gv = m$. In the absence of a θ parameter, the one-loop renormalized coupling is given by

$$\frac{4\pi i}{g^2} = \tau(a) = \frac{\partial^2 \mathcal{F}}{\partial a^2} = \frac{i}{\pi} \left(\ln \frac{a^2}{\Lambda^2} + 3 \right). \quad (7.6)$$

This definition agrees with the on-shell renormalization scheme that we have considered above, because the latter involves only the zero-momentum limit of the two-point function of the massless fields. For a single magnetic monopole, the central charge is given by

$$|U| = |a_D| = \left| \frac{\partial \mathcal{F}}{\partial a} \right| = \frac{1}{\pi} a \left(\ln \frac{a^2}{\Lambda^2} + 1 \right) = \frac{4\pi a}{g^2} - \frac{2a}{\pi}, \quad (7.7)$$

and since $a = m$, this exactly agrees with the result of our direct calculation in (7.4).

The low-energy effective action associated with (7.5) has been derived from a consistency requirement with the anomaly of the $U(1)_R$ symmetry of the microscopic theory, which forms a multiplet with the trace anomaly and a new anomaly in the conformal central charge, which is responsible for the nonzero correction (7.4).

²⁴We are grateful to Horatiu Nastase for pointing this out to us.

	bulk		surf. terms	ordinary c.t.	comp.op. renorm.	total result
	nonan.	anomalous				
$N = 2, M^{(1)}$	xI	$-2m/\pi$	0	$-xI$	0	$-2m/\pi$
$N = 2, U^{(1)}$	0	$-2m/\pi$	xI	$-xI$	0	$-2m/\pi$
$N = 4, M^{(1)}$	0	0	xI	0	$-xI$	0
$N = 4, U^{(1)}$	0	0	xI	0	$-xI$	0

Table 1: Individual contributions to the one-loop corrections to mass ($M^{(1)}$) and central charge ($U^{(1)}$) of $N = 2$ and $N = 4$ monopoles. Here $x = -16\pi m$ and I is the divergent integral defined in (3.21).

8. Conclusions

In this article we have calculated the one-loop corrections to the mass and central charge of $N = 2$ and $N = 4$ susy monopoles in 3+1 dimensions. Besides the nonvanishing anomalous contribution in the $N = 2$ case missed in the older literature, the calculation in the $N = 4$ case involved two novel features: surface terms contributing to the mass corrections and composite operator renormalization of both the mass and the central charge. With everything taken into account, we explicitly verified BPS saturation at the quantum level.

Table 1 summarizes our findings by listing the individual contributions to the one-loop corrections to mass and central charge of the $N = 2$ and $N = 4$ monopoles. The mass contributions involve “bulk” terms which can be identified as the familiar sums over zero-point energies. These cancel in $N = 4$, but in $N = 2$ give a divergent contribution containing a finite anomalous contribution that is left over after standard coupling constant renormalization (denoted by “ordinary c.t.”). As we have discussed above, in our scheme of dimensional regularization this anomalous contribution appears as a remainder of parity violation in the fifth dimension, which is in turn caused by the fact that the fermionic mode functions χ_k^+ and χ_k^- in (3.16) come with normalization factors $\sqrt{\omega + \ell}$ and $\sqrt{\omega - \ell}$, respectively, where ℓ is the momentum of modes along the fifth dimension. This asymmetry is made possible by the existence of two inequivalent representations of the Dirac algebra in 5 dimensions and leads to a net momentum flow $\langle T_{05} \rangle$ which in 3+1 dimensions becomes the anomalous contribution to the central charge. Just as the trace anomaly in T_μ^μ is the anomaly in the conservation of the conformal stress tensor $x^\nu T_{\nu\mu}$, and $\gamma \cdot j_{\text{susy}}$ is the anomaly in the conformal susy current $\not{x} j_{\text{susy}}$, also U_{anom} is the anomaly in the conformal central charge current, in perfect analogy to the situation in the susy kink [24].

The nonanomalous part of the central charge is given by the usual surface term in 3+1 dimensions. Its one-loop correction is divergent and is cancelled by the counterterm induced by standard coupling constant renormalization in the case of $N = 2$, but in the finite $N = 4$ theory, it is cancelled by infinite composite operator renormalization as we have shown in detail in ref. [40]. In the present paper we have shown that there are also surface terms in the mass formula, due to partial integration of the bosonic terms of the form $\partial\varphi\partial\varphi$ in the gravitational stress tensor, while the usual sum over zero-point energies corresponds to bulk terms of the form $-\varphi\partial^2\varphi$. The surface terms $\partial(\varphi\partial\varphi)$ give divergences that precisely match the composite operator renormalization of the susy current multiplet.

The need for composite operator renormalization may be a little surprising. However, nonrenormalization theorems for conserved currents are restricted to internal symmetries (like the vector current in the CVC theory), and do not hold in general for space-time symmetries. The energy-momentum tensor generally requires nonmultiplicative renormalization through improvement terms [57–61]. Curiously, in the $N = 2$ theory both the unimproved and the improved stress tensor is finite (i.e., does not require composite operator renormalization), whereas in the “finite” $N = 4$ theory only the improved stress tensor is finite. The standard (textbook) result for the classical monopole mass refers to the unimproved stress tensor, which is also what is obtained by dimensional reduction from higher dimensions. In 3+1 dimensions, one could also start with an improved stress tensor, and this would also make the $N = 4$ theory free from composite operator renormalization of the susy current multiplet. Note that this reduces the classical value of the mass of a monopole to $2/3$ of its standard value [40].

It would be interesting to study the complete structure of the multiplets of the local improved and nonimproved currents, both in x -space and in superspace. In this connection it may be relevant to note that the stress tensor one obtains in the susy multiplet of currents differs from the gravitational stress tensor by a total derivative.

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